Graduate String Theory: First Principles to Dualities and Beyond

This full graduate-level course provides a complete, first-principles introduction to string theory for students with a solid undergraduate physics background (classical mechanics, quantum mechanics, special relativity, electromagnetism, statistical mechanics, and introductory modern/particle physics). It builds systematically from classical relativistic strings through quantization, conformal field theory, interactions, superstrings, dualities, D-branes, compactifications, M-theory, and modern applications such as AdS/CFT and string phenomenology, emphasizing physical mechanisms and active reconstruction of ideas in the spirit of Richard Feynman. Drawing on standard university structures (e.g., MIT OCW 8.821/8.251, Cambridge Part III courses, and classic texts like Polchinski, Tong, Zwiebach, Becker-Becker-Schwarz, and Green-Schwarz-Witten), the 12 modules develop calculational mastery alongside conceptual depth.

The curriculum prioritizes understanding the 'thing' over mere names: deriving actions and spectra from reparametrization invariance and consistency, reconstructing critical dimensions from Lorentz invariance or Weyl anomalies, and seeing how dualities emerge as equivalences of descriptions. Prerequisites missing from typical solid undergrad training—deeper path-integral QFT techniques and curved-spacetime GR tools—are covered in the opening modules. Each subsequent module builds directly on prior ones, progressing from free bosonic strings to the web of dualities unifying the five superstring theories and 11d supergravity/M-theory. By the end, students can compute spectra, amplitudes, and low-energy effective actions; analyze T-duality and D-brane dynamics; and connect string theory to black-hole thermodynamics, holography, and particle-physics model building.

Assessment and learning emphasize problem-solving (reconstructing derivations from memory after guided reading), with searchHints pointing to primary lecture notes, arXiv reviews, and OCW materials for self-study. The course equips learners for research-level work or further specialization in quantum gravity, holography, or string phenomenology.

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Contents

1 Essential Quantum Field Theory and Path Integrals for Strings

1.1 Canonical versus Path-Integral Quantization

A free real scalar field in Minkowski space can be viewed as an infinite collection of harmonic oscillators. In the canonical approach one expands the field in Fourier modes,

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right),$$

with $\omega_p = \sqrt{p^2 + m^2}$. The equal-time commutation relations

$$[\varphi(x), \pi(y)] = i\delta^3(x - y)$$

translate into $\left[a_p, a_q^\dagger\right] = (2\pi)^3 2\omega_p \delta^3(p - q)$. The vacuum is defined by

$$a_p \ |0 >= 0,$$

and normal ordering subtracts the divergent zero-point energy. This construction yields a Fock space whose states are interpreted as particles.

The path-integral formulation instead defines the vacuum-to-vacuum amplitude directly as a functional integral over all field configurations. After a Wick rotation to Euclidean signature the integral converges and correlators are obtained by $< T\varphi(x_1)...\varphi(x_n) >= \left(\frac{1}{Z}\right) \int D\varphi, \varphi(x_1)...\varphi(x_n) \exp\left(-S_{E[\varphi]}\right)$. The two routes are equivalent for free fields: the mode expansion of the canonical theory reproduces the same two-point function that emerges from the Gaussian path integral. The path-integral language, however, extends immediately to interacting theories and to theories with local symmetries.

1.2 Functional Integrals, Generating Functionals, and Wick Contractions

Introduce an external source J and define the generating functional $Z[J] = \int D\varphi, \exp(iS[\varphi] + i \int J\varphi)$. All time-ordered correlation functions follow by functional differentiation: $< T\varphi(x_1)...\varphi(x_n) >= (-i)^n \left(\delta^n \frac{Z}{\delta} J(x_1)...\delta J(x_n)\right) |_{J=0}$. For the free scalar the action is quadratic. Completing the square yields the exact result

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2} \int J(x) \Delta_{F(x-y)} J(y) d^4x d^4y\right),$$

where

$$\Delta_F$$

is the Feynman propagator satisfying $(\Box + m^2)\Delta_F = -i\delta^4(x - y)$. Repeated differentiation reproduces Wick's theorem: every correlator is the sum over all possible pairings, each pairing replaced by a propagator. The same Gaussian integral, after analytic continuation, supplies the Euclidean propagator used in world-sheet calculations.

1.3 Gauge Fixing, Faddeev-Popov Ghosts, and BRST Symmetry

When the action is invariant under a local gauge transformation

$$A_\mu \to A_\mu + D_\mu \Lambda,$$

the path integral overcounts physically equivalent configurations. The Faddeev-Popov procedure inserts the identity

$$1 = \int D\Lambda, \delta\big(f\left[A^{\Lambda}\right]\big) \det\left(\delta\frac{f}{\delta}\Lambda\right),$$

where

$$f[A] = 0$$

defines a gauge slice (for example the Lorentz condition

$$\partial^{\mu} A_{\mu} = 0$$

). The functional determinant is represented by a pair of anticommuting ghost fields

$$c^{a}$$

and

$$\mid c^{a} :$$

$$\det(M) = \int D \mid cDc, \exp\left(i \int \mid c^{a} M^{a}_{b} c^{b}\right),$$

with $M^{a}_{b} = \partial^{\mu} D^{\mathrm{ab}}_{\mu}$. Adding the gauge-fixing term

$$\frac{\left(\partial^{\mu} A_{\mu}\right)^{2}}{2\xi}$$

produces the gauge-fixed action plus the ghost action $S_{\text{ghost}} = \int \mid c^{a}\left(\partial^{\mu} D_{\mu} c\right)^{a} d^{4}x$. The resulting theory is no longer gauge invariant but is invariant under a global fermionic symmetry called BRST. The BRST charge is nilpotent,

$$Q^{2} = 0,$$

and physical states are identified with its cohomology. On the string world-sheet the same construction removes the reparametrization redundancy and supplies the

$$bc$$

ghost system whose central charge cancels that of the matter fields.

1.4 One-Loop Effective Actions and Conformal Invariance in Two Dimensions

The one-loop correction to the effective action is controlled by the functional determinant of the quadratic fluctuation operator. For a massless scalar in two Euclidean dimensions the determinant is regularized by zeta-function techniques on a torus or cylinder. The resulting partition function transforms covariantly under modular transformations for the free massless scalar ($c = 1$) after zeta-function regularization on the torus, provided zero-mode and compactification handling are treated correctly. The trace anomaly of the stress tensor,

$$< T^{\mu}_{\mu} >= \left(\frac{c}{12}\right) R,$$

vanishes if and only if $c = 0$. For the free boson the beta function vanishes identically ($c = 1$), so the theory remains conformally invariant at the quantum level (while still exhibiting a c-proportional

Weyl anomaly). World-sheet consistency requires the total central charge (matter + ghosts) to vanish; this forces the spacetime dimension to be 26 (or 10 in the supersymmetric case).

1.5 Spinors and Supersymmetry in Two Dimensions

A (holomorphic) Majorana-Weyl fermion in two dimensions has a single real component. Its (formal) Euclidean action is often written as $S_F = \left(\frac{1}{2}\right) \int \psi \partial \psi$. The propagator/OPE is $< \psi(z)\psi(w) >= \frac{1}{z-w}$. Adding a free boson and a free fermion produces a theory whose total central charge is $c = 1 + \frac{1}{2} = \frac{3}{2}$. The supercurrent

$$G(z) = \psi \partial X + ...$$

generates a superconformal algebra whose OPEs close without anomaly precisely when the matter and ghost contributions to the central charge cancel. In ten dimensions the same counting, now with Majorana-Weyl spinors of definite chirality, yields the critical dimension of the superstring. The two-dimensional supersymmetry algebra is therefore the direct ancestor of the higher-dimensional super-Poincaré algebra that appears in the target-space effective theory.

1.6 Sources

- MIT 18.238 Geometry and Quantum Field Theory, Spring 2023
- MIT 8.323 Relativistic Quantum Field Theory I, Spring 2023
- MIT 8.324 Relativistic Quantum Field Theory II, Fall 2010, Lecture 5
- Gradient formula for beta functions in two-dimensional quantum field theory

1.7 Exercises

Exercise 1. Explain, in one paragraph, why the path-integral formulation is immediately applicable to interacting theories and to theories possessing local gauge symmetries, whereas the canonical oscillator approach requires additional machinery.

Exercise 2. Starting from the free generating functional

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2} \int J(x) \Delta_{F(x-y)} J(y) d^4x d^4y\right),$$

perform the functional differentiations needed to obtain the four-point function

$$< T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) >$$

and write the explicit sum of three Wick contractions.

Exercise 3. For a gauge theory with local symmetry

$$A_\mu \to A_\mu + D_\mu \Lambda,$$

insert the Faddeev-Popov identity that enforces the Lorentz gauge $\partial^\mu A_\mu = 0$. Identify the resulting functional determinant and the pair of anticommuting ghost fields that represent it.

Exercise 4. Write the ghost action

$$S_{\text{ghost}}$$

that arises after gauge fixing a non-Abelian Yang-Mills theory to the Lorentz gauge. Show that the full gauge-fixed action is invariant under a nilpotent BRST transformation

$$Q$$

with $Q^2 = 0$.

Exercise 5. A free massless real scalar in two Euclidean dimensions has central charge $c = 1$. Adding a single Majorana-Weyl fermion raises the total central charge to $\frac{3}{2}$. Compute the one-loop trace anomaly

$$< T^\mu_\mu >$$

on a curved world-sheet and state the condition under which it vanishes.

Exercise 6. On the string world-sheet the

$$\mathrm{bc}$$

ghost system contributes central charge $c_{\text{ghost}} = -26$. Determine the number of spacetime dimensions

$$D$$

for which the total (matter + ghost) central charge vanishes when the matter sector consists of

$$D$$

free bosons.

Exercise 7. Consider the one-loop effective action obtained from the functional determinant of the quadratic fluctuation operator of a free scalar on a flat torus. Using zeta-function regularization, show that the resulting partition function is modular invariant precisely when the central charge equals one, and explain why this guarantees the vanishing of the beta function.

Exercise 8. A holomorphic supercurrent on the world-sheet is given by $G(z) = \psi\partial X + ...$ Form the OPE

$$G(z)G(w)$$

and extract the central-charge term. Demonstrate that the superconformal anomaly cancels if and only if the total matter central charge is

$$\frac{3}{2}$$

per superfield, and state the resulting critical dimension for the superstring.

1.8 Solutions

Solution to Exercise 1.

In the canonical formalism the field is expanded in an infinite set of creation and annihilation operators whose commutation relations must be imposed mode by mode; interactions or local redundancies then require a complicated redefinition of the Hilbert space. The path integral, by contrast, directly sums over all field histories weighted by $\exp(iS)$. Gauge orbits are handled by inserting a delta-functional constraint together with the associated Jacobian (Faddeev-Popov determinant), automatically producing the correct measure without ever constructing a physical Hilbert space explicitly. The same functional integral works equally well for non-quadratic actions.

Solution to Exercise 2.

Differentiating

$$Z_0[J]$$

four times with respect to the sources and setting

$$J = 0$$

yields exactly three pairings: $< T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) >= \Delta_{F(x_1-x_2)}\Delta_{F(x_3-x_4)} + \Delta_{F(x_1-x_3)}\Delta_{F(x_2-x_4)} + \Delta_{F(x_1-x_4)}\Delta_{F(x_2-x_3)}$. Each term arises because the exponential is Gaussian; every derivative either hits the exponent (producing a propagator) or differentiates an already-produced propagator (which vanishes at

$$J = 0$$

).

Solution to Exercise 3.

The identity

$$1 = \int D\Lambda, \delta\big(\partial^\mu A_\mu^\Lambda\big) \det\left(\frac{\delta(\partial A^\Lambda)}{\delta}\Lambda\right)$$

is inserted into the path integral. The functional determinant is represented by Grassmann integration over ghost fields

$$\mid c$$

and

$$c,$$

$$\det(M) = \int D \mid cDc, \exp\left(i \int \mid c\big(\partial^\mu D_\mu\big)c\right),$$

where

$$M = \partial^\mu D_\mu$$

is the gauge-field-dependent operator obtained by varying the gauge condition.

Solution to Exercise 4.

The ghost action is $S_{\text{ghost}} = \int d^4x, \mid c^a\big(\partial^\mu D_\mu^{\text{ab}}\big)c^b$. The BRST transformations are

$$QA_\mu^a = D_\mu^{\mathrm{ab}} c^b, \quad Qc^a = -\frac{1}{2} f^{\mathrm{abc}} c^b c^c, \quad Q \mid c^a = b^a,$$

with

$$b^a$$

the Nakanishi-Lautrup field. Direct substitution shows that every term in the gauge-fixed action changes by a total derivative or by a term proportional to the gauge-fixing condition, hence the action is BRST closed. Because

$$Q^2$$

annihilates every field, the charge is nilpotent.

Solution to Exercise 5.

The trace anomaly reads $< T_\mu^\mu >= (\frac{c}{12})R$. With one boson (

$$c = 1$$

) and one Majorana-Weyl fermion (

$$c = \frac{1}{2}$$

) the coefficient is $\frac{3}{2}$. The anomaly therefore vanishes if and only if

$$c = 0,$$

which is impossible for this field content; conformal invariance on the world-sheet is recovered only after the ghost contribution cancels the total matter central charge.

Solution to Exercise 6.

Each free boson contributes

$$c = 1,$$

so

$$D$$

bosons give central charge D. The

$$\mathrm{bc}$$

ghosts contribute -26. Setting the sum to zero yields $D - 26 = 0 \Rightarrow D = 26$. (The same counting with superghosts and fermions replaces 26 by 10.)

Solution to Exercise 7.

The zeta-regularized determinant on the torus produces a factor

$$|\eta(\tau)|^{-2}$$

for a single free boson. Under the modular transformation

$$\tau \to -\frac{1}{\tau}$$

the Dedekind eta function transforms with a phase that cancels precisely when the central charge is unity. Because the partition function is then invariant under all large diffeomorphisms, the Weyl anomaly (which would generate a non-zero beta function) must vanish; hence the theory remains conformally invariant at one loop.

Solution to Exercise 8.

The OPE of two supercurrents contains the term $G(z)G(w) \sim \left(2\frac{c}{3}\right)(z-w)^{-3} + ...$ For each superfield the matter contribution is $c = \frac{3}{2}$. Adding the superghost system (central charge

$$-\frac{15}{2}$$

per pair) produces exact cancellation of the

$$(z-w)^{-3}$$

pole if and only if ten such superfields are present. The resulting critical dimension of the superstring is therefore ten.

2 Essential General Relativity, Differential Geometry, and Supergravity Preliminaries

2.1 The Einstein-Hilbert Action and Einstein Equations

The Einstein field equations follow from a variational principle applied to the Einstein-Hilbert action. In the absence of matter the action is proportional to the integral of the Ricci scalar constructed from the metric and its derivatives. Variation of this action with respect to the metric yields the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - (\frac{1}{2})g_{\mu\nu}R$. Setting the variation to zero produces the vacuum equations $G_{\mu\nu} = 0$. When matter is present the same procedure supplies the source term, giving the conventional form

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

in units where $G = c = 1$.

The resulting equations are second-order and automatically invariant under diffeomorphisms. Linearized gravity is recovered by expanding the metric about a flat background, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h| \ll 1$. Substituting into the Einstein-Hilbert action and retaining quadratic terms produces a wave equation for $h_{\mu\nu}$ whose static, weak-field limit reproduces the Newtonian potential.

Key Idea

The variational origin of the Einstein equations guarantees that the theory is diffeomorphism invariant and that the equations are second order in derivatives of the metric.

2.2 Black-Hole Solutions and Thermodynamics

The Schwarzschild geometry is the unique spherically symmetric vacuum solution. Its event horizon is a Killing horizon whose surface gravity κ determines the temperature measured by asymptotic observers: $T_{\text{BH}} = \frac{\kappa}{2\pi}$. For a black hole of mass M this temperature evaluates to

$$T_{\text{BH}} = \frac{1}{8\pi M}$$

(in natural units). The same horizon area A enters the Bekenstein-Hawking entropy $S = \frac{A}{4}$. These quantities satisfy the first law of black-hole mechanics,

$$dM = \frac{\kappa}{8\pi} dA + ...,$$

which parallels the ordinary thermodynamic relation $dE = TdS$.

Reissner-Nordström solutions are mentioned in passing in thermodynamic contexts; explicit metric forms or thermodynamic extraction beyond the Schwarzschild case are not detailed. Temperature is always measured by static observers at infinity; the local notion of temperature is furnished by the surface gravity itself.

Theorem — Area law

Bekenstein-Hawking entropy is proportional to the horizon area ($S = \frac{A}{4}$ in Planck units); thermodynamic quantities (temperature, entropy) satisfy the first law of black-hole mechanics.

2.3 Supergravity Multiplets in Ten and Eleven Dimensions

Eleven-dimensional maximal supergravity is the highest dimension in which a consistent interacting theory with spins no higher than two can be formulated. Its massless supermultiplet comprises three fields whose on-shell degrees of freedom (counted with the little group SO(9)) are

- graviton G_{MN} (44 bosonic),
- gravitino ψ_M (128 fermionic),
- three-form gauge field C_{KMN} (84 bosonic).

The total is 128 bosonic plus 128 fermionic degrees of freedom. The three-form transforms under the gauge symmetry $\delta C = d\Lambda$. At the two-derivative level the theory is unique.

Dimensional reduction on a circle yields the massless sector of type IIA supergravity in ten dimensions. The ten-dimensional $N = 1$ supergravity multiplet contains the metric, dilaton, two-form, gravitino and dilatino. Type IIB supergravity is obtained by a different pairing of ten-dimensional multiplets that includes a self-dual four-form. All lower-dimensional maximal supergravities arise by further toroidal compactification, with field content fixed by decomposition of the higher-dimensional multiplets.

Definition — p-form gauge symmetry

A p-form potential C admits the transformation $\delta C = d\Lambda$ where Λ is a $(p-1)$-form. This redundancy is responsible for the correct counting of physical degrees of freedom and survives dimensional reduction.

2.4 Linearized Gravity and Consistency Checks

Expanding the Einstein-Hilbert action to quadratic order around flat space produces the linearized Einstein equations. These equations propagate a massless spin-2 field whose coupling to matter reproduces Newtonian gravity at leading order. The same expansion is the starting point for perturbative studies of gravitational waves and for the weak-field limit of black-hole thermodynamics.

Key Idea

On-shell counting in higher-dimensional supergravity is performed with the little group of the Lorentz group; the equality of bosonic and fermionic degrees of freedom is a direct consequence of supersymmetry.

2.5 Sources

- MIT OCW 8.962 Lecture 13 notes (PDF transcript)
- MIT OCW 8.962 Lecture 13
- MIT OCW 8.962 Lecture 14
- arXiv:2412.16795
- arXiv:2507.03778
- arXiv:2512.24929
- arXiv:2303.12682
- MIT OCW 8.962 companion PDF on linearized gravity
- TASI lectures on supergravity and string vacua

2.6 Exercises

Exercise 1. Explain why the Einstein equations obtained from varying the Einstein-Hilbert action are automatically invariant under diffeomorphisms.

Exercise 2. Starting from the definition $G_{\mu\nu} = R_{\mu\nu} - \left(\frac{1}{2}\right)g_{\mu\nu}R$, show that the vacuum Einstein equations $G_{\mu\nu} = 0$ follow directly from setting the metric variation of the Einstein-Hilbert action to zero.

Exercise 3. For a Schwarzschild black hole of mass M, compute the Hawking temperature T_{BH} measured by a static observer at infinity, expressing the result in natural units.

Exercise 4. Derive the Bekenstein-Hawking entropy S of a Schwarzschild black hole in terms of its horizon area A, and verify that the first law of black-hole mechanics reduces to $dM = TdS$ when only mass variations are considered.

Exercise 5. State the on-shell degrees of freedom carried by each field in the massless supermultiplet of eleven-dimensional supergravity and confirm that the total number of bosonic and fermionic degrees of freedom match.

Exercise 6. Explain how the gauge symmetry $\delta C = d\Lambda$ of the three-form in eleven-dimensional supergravity ensures the correct counting of physical degrees of freedom for C_{KMN}.

Exercise 7. Describe the field content obtained by dimensional reduction of eleven-dimensional supergravity on a circle to the massless sector of type IIA supergravity in ten dimensions.

Exercise 8. In the linearized gravity expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h| \ll 1$, argue why the quadratic Einstein-Hilbert action yields a wave equation whose static limit reproduces the Newtonian potential, and relate this to the consistency of the weak-field black-hole thermodynamics discussed in the module.

2.7 Solutions

Solution to Exercise 1.

Diffeomorphism invariance follows because the Einstein-Hilbert action is constructed as a scalar integral $\int R\sqrt{-g}\mathrm{d}^4x$. Any infinitesimal coordinate transformation generated by a vector field ξ^μ changes the metric by a Lie derivative $\mathcal{L}_\xi g_{\mu\nu}$, which is a pure gauge transformation. Since the action is a scalar, its variation under this transformation vanishes identically; therefore the Euler-Lagrange equations (the Einstein tensor set to zero or equal to matter) are automatically invariant under the same diffeomorphisms.

Solution to Exercise 2.

The Einstein-Hilbert action (in units $16\pi G = 1$) is $S = \int R\sqrt{-g}\mathrm{d}^4x$. Its metric variation yields $\delta S = \int G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\mathrm{d}^4x +$ boundary terms. Setting the coefficient of the arbitrary variation

$\delta g^{\mu\nu}$ to zero immediately produces the vacuum field equations $G_{\mu\nu} = 0$. When matter is included the same procedure supplies the source term on the right-hand side, giving the conventional Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$.

Solution to Exercise 3.

The surface gravity of the Schwarzschild horizon is $\kappa = \frac{1}{4M}$. The Hawking temperature measured at infinity is therefore

$$T_{\mathrm{BH}} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M},$$

which is the result quoted in the module.

Solution to Exercise 4.

The horizon area of a Schwarzschild black hole is $A = 4\pi(2M)^2 = 16\pi M^2$. The Bekenstein-Hawking entropy is defined by $S = \frac{A}{4}$, hence $S = 4\pi M^2$. Differentiating gives $dS = 8\pi M dM$. Substituting the Hawking temperature $T = \frac{1}{8\pi M}$ into TdS recovers exactly dM, confirming the first law $dM = TdS$ when angular momentum and charge variations are absent.

Solution to Exercise 5.

The eleven-dimensional graviton G_{MN} carries 44 on-shell degrees of freedom under the little group SO(9), the gravitino ψ_M carries 128 fermionic degrees of freedom, and the three-form C_{KMN} carries 84 bosonic degrees of freedom. Summing the bosonic contributions yields $44 + 84 = 128$, which exactly matches the 128 fermionic degrees of freedom, as required by supersymmetry.

Solution to Exercise 6.

A three-form potential in eleven dimensions has $\binom{9}{3} = 84$ independent components after accounting for the gauge redundancy $\delta C = d\Lambda$. The transformation removes unphysical longitudinal modes, leaving precisely the 84 physical degrees of freedom that enter the on-shell counting of the supermultiplet.

Solution to Exercise 7.

Dimensional reduction on a circle decomposes the eleven-dimensional fields into ten-dimensional fields: the metric yields the ten-dimensional metric, a Kaluza-Klein vector and a dilaton; the three-form yields a three-form and a two-form in ten dimensions; the gravitino decomposes into a gravitino and a dilatino. The resulting massless spectrum is precisely the $N = 2$ (type IIA) supergravity multiplet in ten dimensions.

Solution to Exercise 8.

Substituting the linearized metric into the Einstein-Hilbert action and retaining the quadratic terms produces the Fierz-Pauli action for a massless spin-2 field. Its equations of motion reduce to the wave equation $\Box\bar{h}_{\mu\nu} = 0$ (in the harmonic gauge). In the static, weak-field limit the solution for a point mass reproduces the Newtonian potential $\Phi = -\frac{M}{r}$. Because the same linearized metric describes the asymptotic region of a Schwarzschild black hole, the thermodynamic relations extracted from surface gravity and horizon area remain consistent with the weak-field limit of the Einstein equations.

3 Motivations for String Theory and the Classical Relativistic String

Strings appear across widely separated physical scales. At the largest distances they describe cosmic strings formed in early-universe phase transitions. At the scale of strong interactions they model QCD flux tubes whose tension is fixed by the string tension parameter to reproduce the linear Regge trajectories of mesons, with a typical radius of order 0.2 fm. At still shorter distances they furnish a candidate ultraviolet completion of gravity in which the fundamental objects are one-dimensional rather than point-like. Lecture 1 of MIT OCW 8.251 emphasizes precisely this multiplicity of realizations while contrasting the situation with the Standard Model, whose point-particle fields leave gravity non-renormalizable.

The central ultraviolet difficulty of point-particle quantum gravity is that the Einstein-Hilbert action yields a dimensionful coupling whose negative mass dimension produces an ever-growing number of counterterms at each loop order. Replacing the world-line of a point particle by a two-dimensional world-sheet spreads the interaction over a finite region and thereby improves the short-distance behavior. Historically this improvement was first noticed in the dual-resonance model of hadronic scattering, where an infinite tower of resonances lying on linear Regge trajectories emerged automatically once the scattering amplitude was written in Veneziano form. That amplitude is reproduced by the spectrum of a relativistic string, furnishing the original phenomenological motivation for the theory.

3.1 The Nambu-Goto Action

The simplest geometric action for a relativistic string is the proper area of its world-sheet. Parametrizing the embedding by coordinates $X^\mu(\tau,\sigma)$ in a background metric $g_{\mu\nu}(x)$, the induced metric on the world-sheet is $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. The Nambu-Goto action is therefore the integral of the area element,

$$S_{\mathrm{NG}} = -\frac{T_0}{c}\int \mathrm{d}\tau\,\mathrm{d}\sigma\sqrt{-\det h},$$

where T_0 is the string tension. In flat space the determinant expands to the familiar square-root expression given in the 8.251 final examination, $S_{\mathrm{NG}} = -\frac{T_0}{c}\int \mathrm{d}\tau\,\mathrm{d}\sigma\sqrt{\left(\dot{X}\cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}$. Reparametrization invariance is manifest: any monotonic redefinition of τ and σ leaves the action unchanged because it merely changes the coordinate chart on the same geometric surface.

3.2 Gauge Fixing, Equations of Motion, and Mode Expansions

The diffeomorphism and Weyl symmetries allow us to reach the conformal gauge. The action then reduces to a free-field action whose Euler-Lagrange equations are simply the two-dimensional wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. Residual conformal freedom is fixed by the light-cone gauge condition $X^+ = \tau$ (up to normalization). The remaining transverse coordinates satisfy the wave equation subject to boundary conditions that distinguish open and closed strings.

For an open string with Neumann conditions at both ends the general solution is

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma),$$

with the zero-mode identification $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$. For a closed string the periodicity requirement produces two independent sets of oscillators,

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right),$$

where now $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. In both cases the Virasoro constraints $T_{ab} = 0$ reduce to the pair of conditions

$$\dot{X}\cdot X' = 0, \quad \dot{X}^2 + (X')^2 = 0,$$

which become statements about the vanishing of the Fourier modes $L_n = 0$ once the expansions are substituted.

3.3 Classical Conserved Charges and Regge Trajectories

The Noether currents associated with Poincaré invariance yield the total energy-momentum

$$P^\mu = T_0 \int_0^\pi \dot{X}^\mu \, \mathrm{d}\sigma$$

and the angular-momentum tensor $M^{\mu\nu} = T_0 \int_0^\pi \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \mathrm{d}\sigma$. Evaluating these charges on a rigidly rotating open string whose ends move at the speed of light produces the classical Regge relation

$$J = \alpha' M^2,$$

where the Regge slope is related to the tension by $\alpha' = \frac{1}{2\pi T_0}$. No additive intercept appears at the classical level; the intercept arises only after quantization. This linear relation between spin and mass squared was the original phenomenological signature that pointed toward a string description of the strong interaction.

3.4 Sources

- MIT OCW 8.251 Lecture 1 (motivations at multiple scales, QCD flux tubes, fundamental strings)
- MIT OCW 8.251 Final Examination 2007 (Nambu-Goto action, light-cone mode expansions, classical Regge relation $J = \alpha' M^2$)
- MIT OCW 8.821 Lecture materials on light-cone gauge and string actions

3.5 Exercises

Exercise 1. Explain in one paragraph why the negative mass dimension of Newton's constant in four-dimensional Einstein gravity leads to non-renormalizability, and how replacing point particles by extended strings is expected to improve the short-distance behavior.

Exercise 2. State the historical connection between the Veneziano amplitude, linear Regge trajectories, and the spectrum of a relativistic string. What phenomenological feature of meson resonances first suggested a string description of the strong interaction?

Exercise 3. Starting from the induced metric on the world-sheet, $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, derive the Nambu-Goto action by requiring that it equal (minus) the string tension times the proper area of the embedded surface.

Exercise 4. In the conformal gauge the Nambu-Goto action reduces to a free-field action. Write the resulting Euler-Lagrange equation for each embedding coordinate $X^{\mu(\tau,\sigma)}$ and state the two residual Virasoro constraints that must still be imposed.

Exercise 5. Write the most general solution of the two-dimensional wave equation that satisfies Neumann boundary conditions at both ends of an open string of length π. Identify the zero-mode term with the center-of-mass momentum and express the normalization constants in terms of α'.

Exercise 6. For a closed string, periodicity in σ implies two independent sets of oscillators. Write the mode expansion and the corresponding relation between α_0^μ and the total momentum p^μ.

Exercise 7. Using the Noether procedure for translations, derive the total conserved four-momentum P^μ of an open string in terms of an integral over $\dot{X}^\mu$. Then compute the angular-momentum tensor $M^{\mu\nu}$ from the same procedure applied to Lorentz transformations.

Exercise 8. Consider a rigidly rotating open string whose endpoints move at the speed of light. Evaluate the classical Regge relation between angular momentum J and mass squared M^2 and show that it takes the form $J = \alpha' M^2$ with $\alpha' = \frac{1}{2\pi T_0}$.

3.6 Solutions

Solution to Exercise 1.

In the Einstein-Hilbert action the gravitational coupling G_N has engineering dimension of inverse mass squared. Each additional loop therefore brings an extra factor of G_N accompanied by an integral whose ultraviolet divergence grows with the loop order, forcing an infinite tower of higher-curvature counterterms. Because a string has finite extent, its world-sheet interaction is smeared over a region of size $\sqrt{\alpha'}$, cutting off the momentum integrals at $\Lambda \sim \frac{1}{\sqrt{\alpha'}}$ and rendering the theory ultraviolet finite order by order in perturbation theory.

Solution to Exercise 2.

The Veneziano amplitude is a four-point scattering function whose poles lie on linear trajectories $J = \alpha' m^2 + \alpha_0$. The same linear relation between spin and mass squared is reproduced exactly by the spectrum of a classical relativistic string. This match with the observed Regge trajectories of mesons provided the original phenomenological motivation for string theory as a model of QCD flux tubes.

Solution to Exercise 3.

The proper area element on the world-sheet is $dA = \sqrt{-\det h}\, \mathrm{d}\tau\, \mathrm{d}\sigma$. Multiplying by the constant tension T_0 (and restoring c) gives the geometric action $S_{\mathrm{NG}} = -\frac{T_0}{c} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\det h}$, where the induced metric h_{ab} is formed by pulling back the spacetime metric with the embedding functions $X^{\mu(\tau,\sigma)}$. This is the Nambu-Goto action.

Solution to Exercise 4.

After conformal-gauge fixing the action becomes that of D free scalars, $S = -\frac{T_0}{2}\int\left(\dot{X}^2 - (X')^2\right) \mathrm{d}\tau\, \mathrm{d}\sigma$. The Euler-Lagrange equation for each X^μ is therefore the wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$. The two Virasoro constraints that remain are $\dot{X} \cdot X' = 0, \quad \dot{X}^2 + (X')^2 = 0$.

Solution to Exercise 5.

The general solution of the wave equation on the interval $0 \leq \sigma \leq \pi$ with $\partial_\sigma X^\mu = 0$ at the endpoints is $X^{\mu(\tau,\sigma)} = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n=1}^\infty \frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \alpha_n^\mu e^{-in(\tau+\sigma)}\right)$, which is equivalently written with a cosine sum. Identifying the zero-mode coefficient with the center-of-mass momentum gives $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

Solution to Exercise 6.

Periodicity $X^{\mu(\tau,\sigma+2\pi)} = X^{\mu(\tau,\sigma)}$ requires two independent sets of modes: $X^\mu = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right)$. The zero-mode normalization compatible with the total momentum is now $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$.

Solution to Exercise 7.

The spacetime translation $\delta X^\mu = \varepsilon^\mu$ yields the conserved current whose integral is $P^\mu = T_0\int_0^\pi \dot{X}^\mu \, \mathrm{d}\sigma$. The Lorentz transformation $\delta X^\mu = \omega^\mu_\nu X^\nu$ produces the angular-momentum tensor $M^{\mu\nu} = T_0\int_0^\pi \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \mathrm{d}\sigma$.

Solution to Exercise 8.

Parametrize a rotating open string by $X^0 = \tau,\ X^1 = A\cos\sigma\cos\tau,\ X^2 = A\sin\sigma\sin\tau$. The constraints fix $A = \pi\sqrt{\alpha'}$ (endpoints move at light speed). Substituting into the expressions for P^μ and $M^{\mu\nu}$ yields the mass $M = T_0\pi A$ and angular momentum $J = \frac{T_0\pi A^2}{2}$. Eliminating A immediately produces the classical Regge trajectory $J = \alpha' M^2$ with the identification $\alpha' = \frac{1}{2\pi T_0}$.

4 Light-Cone Quantization of the Bosonic String

4.1 Light-Cone Coordinates and Gauge Choice

In the Polyakov formulation the worldsheet metric is fixed to the conformally flat form $\gamma_{ab} = \eta_{ab}$ by a combination of diffeomorphisms and Weyl rescalings. Residual conformal symmetry remains after this step. The light-cone gauge exploits the remaining freedom by aligning the worldsheet time coordinate with a spacetime light-cone direction. In light-cone gauge, $X^+ \equiv \frac{X^0+X^{D-1}}{\sqrt{2}}$ is fixed to $X^+ = x^+ + c\alpha' p^+ \tau$ (where $c = 2$ for open strings in many conventions and the precise factor varies for closed strings; x^+ is the constant mode). This choice completely eliminates the longitudinal oscillators, leaving only the transverse coordinates X^i $(i = 2, ..., D-1)$ as independent dynamical degrees of freedom.

The classical equations of motion that follow from the Polyakov action reduce to the two-dimensional wave equation $\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0$. After imposing the light-cone gauge, the same wave equation holds for each transverse field X^i. Open-string boundary conditions are either Neumann ($\partial_\sigma X^i = 0$ at the endpoints) or Dirichlet; the latter will be reinterpreted below as the presence of D-branes.

4.2 Oscillator Quantization and Normal Ordering

The general solution of the wave equation is expanded in Fourier modes. In light-cone gauge only transverse oscillators survive. For open strings:

$$X^i(\tau, \sigma) = x^i + 2\alpha' p^i \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \left(\frac{1}{n}\right) \alpha_n^i e^{-in\tau} \cos(n\sigma) + ...$$

(closed strings have independent left- and right-moving traveling-wave modes with $\sqrt{\frac{\alpha'}{2}}$ normalization). Canonical quantization promotes the modes to operators obeying the commutation relations $[\alpha_m^i, \alpha_n^j] = m\delta_{m+n,0}\delta^{ij}$. The Hamiltonian and the Virasoro constraints involve normal-ordered products of these oscillators. The zero-point energy that arises from the infinite sum over modes is regularized by zeta-function methods and yields a finite normal-ordering constant a. This constant enters the mass-shell condition and must be determined self-consistently from Lorentz invariance.

Key Idea

The light-cone gauge makes only an $\mathrm{SO}(D-2)$ subgroup of the Lorentz group manifest. Closure of the full $\mathrm{SO}(1, D-1)$ algebra on the quantum spectrum will fix both the intercept a and the spacetime dimension D.

4.3 Mass-Squared Operator and Level Matching

After normal ordering, the mass-squared operator for open strings reads

$$M^2 = \frac{1}{\alpha'}(N - a),$$

where the number operator is $N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i$. Physical states satisfy the Virasoro constraints. In light-cone gauge these reduce to the mass-shell condition $M^2 = \frac{N-a}{\alpha'}$ for open strings, and for closed strings to $M^2 = 2\frac{N+\tilde{N}-2a}{\alpha'}$ together with the level-matching requirement $N = \tilde{N}$.

For closed strings two independent sets of oscillators appear and the mass formula becomes

$$M^2 = \frac{2}{\alpha'}\big(N + \tilde{N} - 2a\big),$$

subject to the level-matching condition $N = \tilde{N}$. The ground state ($N = 0$) is a scalar tachyon with $M^2 = -\frac{a}{\alpha'}$. The first excited level of the open string ($N = 1$) transforms as a vector under the transverse rotation group $\mathrm{SO}(D-2)$.

Definition — Physical Hilbert space

States are built by acting with transverse creation operators on the Fock vacuum. In light-cone gauge the mass-shell condition is imposed via the mass formula ($M^2 = \frac{N-a}{\alpha'}$ open; $M^2 = 2\frac{N+\tilde{N}-2a}{\alpha'}$ closed); level-matching $N = \tilde{N}$ is additionally imposed for closed strings.

4.4 Critical Dimension from Lorentz Algebra Closure

The first excited open-string state is a transverse vector possessing $D-2$ polarization states. For this state to be massless and to transform correctly under the little group of a massless particle, it must belong to the vector representation of $\mathrm{SO}(D-2)$. Consistency with the full Lorentz algebra $\mathrm{SO}(1, D-1)$ then forces the intercept and dimension to the unique values $a = 1, \quad D = 26$. With these values the massless closed-string states likewise assemble into proper representations: a symmetric traceless tensor (graviton), a scalar (dilaton), and an antisymmetric tensor (Kalb-Ramond field). All higher massive levels automatically fill complete $\mathrm{SO}(D-1)$ multiplets once the critical dimension is satisfied.

Theorem — Critical dimension

Lorentz invariance of the quantum spectrum requires $D = 26$ and $a = 1$. This is the only value for which the transverse vector at level one is massless and the algebra of Lorentz generators closes without anomalies.

4.5 Open-String Spectrum on D-Branes

When Neumann boundary conditions are imposed in all directions the open-string spectrum contains a tachyon ground state, a massless vector (gauge field) at the first excited level, and an infinite tower of massive higher-spin states. If Dirichlet conditions are instead chosen in $D-p-1$ transverse directions, the string endpoints are fixed to a p-dimensional hyperplane—the D-brane. The massless open-string modes localized on the brane then consist of a $U(1)$ gauge field living on the worldvolume together with $D-p-1$ scalars that parametrize transverse fluctuations of the brane.

Example

For a p-brane the massless open-string spectrum comprises a $(p+1)$-dimensional $U(1)$ gauge field plus $(D-p-1)$ real scalars corresponding to transverse fluctuations.

4.6 Sources

- MIT 8.821 Lecture 10
- MIT 8.821 Lecture 12
- MIT 8.821 String spectrum overview
- MIT 8.251 review notes

4.7 Exercises

Exercise 1. In light-cone gauge, how many independent transverse coordinates remain as dynamical degrees of freedom for a bosonic string in spacetime dimension D?

Exercise 2. Write the mass-squared operator M^2 for open strings in terms of the number operator N and the normal-ordering constant a. What additional condition must be imposed for closed strings?

Exercise 3. Identify the ground-state mass M^2 for both open and closed strings. What does a negative value imply about the state?

Exercise 4. For an open string in light-cone gauge with $N = 1$, determine the mass M^2 when $a = 1$. How many independent polarization states does this level possess?

Exercise 5. For closed strings satisfying level matching $N = \tilde{N} = 1$, compute M^2 when $a = 1$. Identify the spacetime representations formed by the massless states under the little group.

Exercise 6. A Dp-brane is embedded in 26-dimensional spacetime. How many massless scalar fields appear in the open-string spectrum localized on the brane, and what physical interpretation do they have?

Exercise 7. The first excited open-string state transforms as a vector under $\mathrm{SO}(D-2)$. Show that requiring this state to be massless and to furnish a representation of the massless little group $\mathrm{SO}(D-2)$ forces the intercept to satisfy $a = 1$.

Exercise 8. Explain why closure of the full Lorentz algebra $\mathrm{SO}(1, D-1)$ on the light-cone spectrum is possible only when $D = 26$. What happens to the higher massive levels once this critical dimension is satisfied?

4.8 Solutions

Solution to Exercise 1.

Light-cone gauge fixes the coordinate X^+ completely, removing all longitudinal oscillators. The remaining independent fields are the transverse coordinates X^i with $i = 2, ..., D-1$, giving exactly $D-2$ dynamical degrees of freedom.

Solution to Exercise 2.

For open strings the mass-squared operator is $M^2 = \frac{1}{\alpha'}(N - a)$. For closed strings the corresponding expression is $M^2 = \frac{2}{\alpha'}\big(N + \tilde{N} - 2a\big)$ together with the level-matching condition $N = \tilde{N}$.

Solution to Exercise 3.

The open-string ground state ($N = 0$) has $M^2 = -\frac{a}{\alpha'}$. The closed-string ground state likewise satisfies $M^2 = -2\frac{a}{\alpha'}$. A negative mass squared signals a tachyon, indicating an instability of the perturbative vacuum.

Solution to Exercise 4.

Substituting $N = 1$ and $a = 1$ yields $M^2 = 0$. The state is therefore massless and carries $D - 2 = 24$ independent transverse polarization vectors when $D = 26$.

Solution to Exercise 5.

Level matching forces $N = \tilde{N} = 1$, so $M^2 = 0$. The resulting massless states transform as a symmetric traceless tensor (graviton), an antisymmetric tensor (Kalb-Ramond field), and a scalar (dilaton) under the little group SO(24).

Solution to Exercise 6.

A Dp-brane has $D - p - 1 = 25 - p$ transverse directions subject to Dirichlet conditions. Consequently the open-string spectrum contains exactly $25 - p$ massless scalar fields; each scalar parametrizes a transverse fluctuation of the brane.

Solution to Exercise 7.

The transverse vector at level $N = 1$ must belong to the vector representation of the massless little group $\mathrm{SO}(D - 2)$. This representation exists only when the state is massless, which occurs precisely when $a = 1$. Any other value of a would produce a massive vector that cannot be consistently embedded into a representation of $\mathrm{SO}(1, D - 1)$.

Solution to Exercise 8.

The commutators of the Lorentz generators receive anomalous contributions proportional to $(D - 26)$ and $(a - 1)$. These anomalies vanish simultaneously only for $D = 26$ and $a = 1$. Once the critical dimension is satisfied, every massive level automatically assembles into complete irreducible representations of the full little group $\mathrm{SO}(D - 1)$.

5 Covariant Quantization, Virasoro Algebra, and the Bosonic Spectrum

5.1 Covariant Quantization and Physical-State Conditions

The starting point is the Polyakov action in conformal gauge, $S_P^{c.g.} = -\frac{T}{2}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X$. After fixing diffeomorphisms and Weyl symmetry the equations of motion reduce to the two-dimensional wave equation $\partial^2 X^\mu = 0$. Mode expansions of the embedding coordinates yield oscillators satisfying

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$$

in the mostly-plus Minkowski metric.

Classically the world-sheet energy-momentum tensor is traceless and conserved; its Fourier modes are the Virasoro generators $L_n = \frac{1}{2}\sum_m : \alpha_{n-m} \cdot \alpha_m :$. In the quantum theory these generators are promoted to operators that must annihilate physical states. The Gupta-Bleuler-style conditions read $L_n \, |\psi >= 0 \quad (n \geq 0)$. (The zero-mode condition is usually written

$$(L_0 - a)|\psi >= 0$$

with a normal-ordering constant that will be fixed by consistency.) States obeying these constraints form the physical subspace; any state that fails them is unphysical and is removed from the spectrum.

Definition — Spurious and null states

A spurious state is any vector of the form

$$L_{-n} \, |\chi >$$

with

$$n > 0$$

and

$$|\chi >$$

physical. When the central charge takes its critical value such states are null (zero norm) and decouple from all scattering amplitudes.

5.2 Virasoro Algebra and Central Charge

Direct evaluation of the commutator of two normal-ordered generators produces the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$. The central term arises solely from the normal-ordering constant in the oscillator commutators. Each free boson

$$X^\mu$$

contributes one unit of central charge, so for

$$D$$

coordinate fields one obtains $c = D$. The same algebra governs both open and closed strings (the closed case simply possesses two independent copies, left- and right-moving).

Theorem — Central charge of free bosons

For

$$D$$

free scalar fields the operator-product expansion of the energy-momentum tensor yields central charge $c = D$. Consequently the Virasoro anomaly vanishes if and only if $D = 26$.

5.3 Criticality at Dimension 26

When

$$c = 26$$

the central term in the Virasoro algebra vanishes, allowing the full set of physical-state conditions to be imposed consistently at every mass level. For any other dimension the algebra fails to close on the physical subspace and negative-norm states remain. This is the origin of the critical dimension of the bosonic string.

Key Idea

The requirement that the Virasoro constraints be first-class at the quantum level forces the space-time dimension to be exactly 26. All subsequent consistency conditions (absence of ghosts, modular invariance, etc.) are satisfied only in this dimension.

5.4 Physical Spectrum and the No-Ghost Theorem

States are constructed level by level from the tachyon vacuum. At each level the subspace obeying

$$L_n \, |\psi >= 0$$

(

$$n \geq 0$$

) contains both positive-norm physical states and null states generated by the L_{-n}. The no-ghost theorem asserts that, precisely when

$$D = 26,$$

every physical state has non-negative norm. The proof proceeds by exhibiting an explicit isomorphism between the physical subspace and a manifestly positive-definite Fock space (commonly realized via light-cone coordinates or via the cohomology of the BRST operator). Null states are quotiented out, leaving a unitary Hilbert space.

Theorem — No-ghost theorem (outline)

In the critical theory the Virasoro module generated by the physical-state conditions is free of negative-norm vectors. The physical Hilbert space is therefore unitary.

5.5 Massless Level and BRST Preliminaries

At the massless level of the closed string the surviving states are transverse, traceless, symmetric rank-2 tensors (graviton), an antisymmetric tensor (Kalb-Ramond field), and a scalar (dilaton);

the massless level of the open string contains the photon (vector). Their polarization tensors satisfy the Virasoro conditions and are automatically transverse.

An equivalent and more powerful formulation replaces the Gupta-Bleuler conditions by BRST cohomology. One introduces anticommuting ghost fields whose central charge is exactly -26. The total BRST charge

$$Q = \sum_n : c_n \left(L_n^m + \frac{1}{2} L_n^g \right) :$$

is then nilpotent,

$$Q^2 = 0,$$

if and only if the matter central charge equals 26. Physical states are identified with the cohomology classes of

$$Q$$

(at the appropriate ghost number, conventionally one for open strings); the Virasoro constraints are recovered automatically as $Q|\psi >= 0$.

Key Idea

BRST quantization encodes the Virasoro constraints in the nilpotency of a single operator whose cohomology reproduces the physical spectrum, including the correct counting of massless graviton, dilaton and Kalb-Ramond degrees of freedom.

5.6 Sources

- arXiv:2602.05173
- arXiv:1107.3967
- arXiv:hep-th/0304254
- arXiv:2601.06247
- arXiv:2208.05179

5.7 Exercises

Exercise 1. State the Gupta-Bleuler physical-state conditions imposed on a string state $|\psi >$ in covariant quantization, including the role of the normal-ordering constant a.

Exercise 2. Define a spurious state and explain under what condition such states become null (zero-norm) vectors that decouple from scattering amplitudes.

Exercise 3. Each free scalar field X^μ contributes one unit to the central charge of the Virasoro algebra. What is the resulting central charge c for the bosonic string in D spacetime dimensions, and at what value of D does the Virasoro anomaly vanish?

Exercise 4. Write the Virasoro algebra commutator $[L_m, L_n]$ including the central term. Identify the origin of the central extension and its explicit form for free bosons.

Exercise 5. Starting from the mode expansion and normal-ordering, compute the central charge contributed by a single free boson by evaluating the coefficient of the cubic term in the Virasoro commutator.

Exercise 6. At the massless level of the closed bosonic string, list the three types of physical states that survive the Virasoro constraints and give their spacetime interpretations (graviton, Kalb-Ramond field, dilaton).

Exercise 7. Show that the BRST charge Q is nilpotent ($Q^2 = 0$) if and only if the matter central charge equals 26. Explain how this reproduces the Virasoro constraints via cohomology.

Exercise 8. Using the no-ghost theorem, argue why the physical Hilbert space is unitary precisely at the critical dimension $D = 26$. Outline the isomorphism to a positive-definite Fock space and the role of null states.

5.8 Solutions

Solution to Exercise 1.

The physical-state conditions are $L_n|\psi \geq 0$ for all $n \geq 0$. The zero-mode condition is written $(L_0 - a)|\psi \geq 0$, where the normal-ordering constant a is fixed by consistency requirements (ultimately $a = 1$ for the open string and $a = 2$ for each sector of the closed string). These are the quantum analogs of the classical vanishing of the energy-momentum tensor modes.

Solution to Exercise 2.

A spurious state has the form $L_{-n}|\chi >$ with $n > 0$ and $|\chi >$ physical. When the central charge equals its critical value these states are null, i.e., they have zero norm, and therefore decouple from all physical scattering amplitudes.

Solution to Exercise 3.

Because each of the D coordinate fields contributes one unit, the total central charge is $c = D$. The Virasoro anomaly vanishes precisely when $D = 26$.

Solution to Exercise 4.

The Virasoro algebra reads $[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}$. The central term arises solely from the normal-ordering ambiguity in the oscillator commutators. For D free bosons one finds $c = D$.

Solution to Exercise 5.

The commutator of two normal-ordered Virasoro generators receives a c-number contribution only from the double contractions of the oscillator modes. Each boson produces a term whose coefficient is exactly 1, so a single free boson yields central charge $c = 1$. Summing over D bosons immediately gives $c = D$.

Solution to Exercise 6.

The massless level contains: (i) a transverse traceless symmetric rank-2 tensor (graviton), (ii) an antisymmetric rank-2 tensor (Kalb-Ramond field), and (iii) a scalar (dilaton). All three satisfy the Virasoro conditions and are automatically transverse.

Solution to Exercise 7.

The BRST charge is $Q = \sum_n : c_{n(L_n^m + \frac{1}{2}L_n^g)} :$. Its square Q^2 receives a contribution proportional to the total central charge. The ghost system contributes $c_g = -26$, so $Q^2 = 0$ if and only if the matter central charge equals 26. The condition $Q|\psi \geq 0$ then encodes all Virasoro constraints, and physical states are identified with the cohomology classes of Q.

Solution to Exercise 8.

At $D = 26$ every physical state obeying the Virasoro conditions has non-negative norm. The no-ghost theorem establishes an explicit isomorphism between the physical subspace and a manifestly positive-definite Fock space realized, for example, in light-cone coordinates. All null states generated by the L_{-n} are quotiented out, leaving a unitary Hilbert space. For any other dimension negative-norm states remain and unitarity is lost.

6 Two-Dimensional Conformal Field Theory for Worldsheets

6.1 Conformal Transformations, Stress Tensor, and Ward Identities

In two Euclidean dimensions, conformal transformations are angle-preserving maps generated by holomorphic functions $f(z)$. Under an infinitesimal map $z \mapsto z + \varepsilon(z)$, a field φ of conformal weight h transforms as $\delta_\varepsilon \varphi = \varepsilon \partial \varphi + h(\partial \varepsilon)\varphi$. The generator of these transformations is the holomorphic stress tensor $T(z)$. Its contour integral around a point w yields the infinitesimal variation: $\delta_\varepsilon \varphi(w) = \frac{1}{2\pi i} \int_C \varepsilon(z) T(z) \varphi(w) \, \mathrm{d}z$. This contour integral encodes the conformal Ward identity. When the contour is deformed around all other operator insertions, the same identity determines the transformation law of any correlation function.

The stress tensor itself is not a primary field. Its self-OPE takes the universal form

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots,$$

where the coefficient c is the central charge. The normalization is chosen so that a single free boson has $c = 1$.

Key Idea

The infinite-dimensional Virasoro algebra arises because every holomorphic vector field $\varepsilon(z) = \sum \varepsilon_n z^{n+1}$ generates a symmetry. The central term proportional to c measures the failure of T to be primary and ultimately controls the critical dimension of string theory.

6.2 Operator Product Expansions and Primary Fields

A field $\varphi_{h(z)}$ is primary of weight h when its OPE with the stress tensor is $T(z)\varphi_{h(w)} \sim \frac{h}{(z-w)^2}\varphi_{h(w)} + \frac{1}{z-w}\partial \varphi_{h(w)} + \dots$. All higher poles are absent. The double pole coefficient encodes the holomorphic conformal weight h while the simple pole generates translations via the derivative term. Descendant fields are obtained by acting with modes of T; their OPEs with T contain additional regular terms that are fixed by the algebra.

The OPE of two primaries determines the fusion rules of the theory. Expanding the product in a basis of fields and taking the singular terms yields the structure constants of the operator algebra. Because the Ward identity fixes the action of T on every descendant, it is sufficient to know the OPEs among primaries.

Definition — Conformal weight

The number h appearing in the leading pole of $T(z)\varphi(w)$ is the holomorphic conformal weight. For a spinless scalar the anti-holomorphic weight $|h|$ equals h, so the scaling dimension is $\Delta = 2h$.

6.3 Virasoro Algebra, Representations, and Characters

Mode expansion of the stress tensor on the cylinder (or plane after radial quantization) produces the Virasoro generators $L_n = \frac{1}{2\pi i} \int z^{n+1} T(z) \, \mathrm{d}z$. Their commutation relations read $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}$. Highest-weight representations are labeled by a pair (c, h). A primary state $|h\rangle$ satisfies $L_n|h\rangle = 0$ for all $n > 0$ and $L_0|h\rangle = h|h\rangle$.

For $c < 1$ unitary representations exist only at the discrete values of the minimal models, where the Kac formula gives the degenerate weights $h_{r,s} = \frac{((m+1)r-ms)^2-1}{4m(m+1)}$. At these points null vectors appear, implying differential equations that correlators must obey. The character of a representation is the graded trace

$$\chi_{h(q)} = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}},$$

which enters the torus partition function after summing over the spectrum with appropriate modular invariance.

Theorem — Central charge of free fields

A free massless boson has central charge $c = 1$. A Majorana fermion contributes $c = \frac{1}{2}$. The bc ghost system with weights $(2, -1)$ yields $c = -26$, precisely cancelling the matter contribution in the critical bosonic string.

6.4 State-Operator Correspondence and Vertex Operators

Radial quantization identifies the vacuum at the origin with the state $|0\rangle$. Inserting an operator $\varphi(z)$ at $z = 0$ creates the state $|\varphi\rangle = \varphi(0)|0\rangle$. Conversely, every state in the Hilbert space corresponds to a unique local operator. This bijection converts questions about string spectra into questions about operator dimensions.

Vertex operators for the bosonic string are normal-ordered exponentials

$$V(k, z) =: \exp(ik \cdot X(z)) :,$$

where the momentum k is chosen so that the total conformal weight equals one. For the tachyon $k^2 = 4$ (in units $\alpha' = 1$); for massless vectors a polarization factor $\varepsilon \cdot \partial X$ multiplies the exponential. BRST invariance further restricts the allowed states, reproducing the light-cone spectrum.

Example

The massless graviton vertex operator in 26 dimensions is

$$V = \varepsilon_{\mu\nu} : \partial X^\mu |\partial| X^\nu \exp(ik \cdot X) :$$

with $k^2 = 0$ and $k \cdot \varepsilon = 0$. Its weight-$(1, 1)$ property follows directly from the free-boson OPE of ∂X with the exponential.

6.5 Free-Field Realizations and Ghost Systems

The free boson φ with action $\frac{1}{8\pi} \int (\partial\varphi)^2$ has OPE $\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{(z-w)^2}$. Its $U(1)$ current $J = i\partial\varphi$ generates a level-1 Kac-Moody algebra. A free Majorana fermion ψ satisfies

$$\psi(z)\psi(w) \sim \frac{1}{z - w},$$

For a complex Dirac fermion ($c = 1$), bosonization identifies $\psi \sim: \exp(i\varphi) :$ (up to cocycle factors); a free Majorana fermion ($c = \frac{1}{2}$) requires a more subtle construction involving square-root or Ising disorder operators.

Gauge fixing the string introduces the bc system with weights $(2, -1)$. The central charge $c = -26$ is required for cancellation against 26 bosons. The superstring retains the bc system ($c = -26$)

along with a $\beta\gamma$ system of weights $\left(\frac{3}{2}, -\frac{1}{2}\right)$ whose central charge is $+11$; the total ghost central charge is -15 and is cancelled by $D = 10$ matter ($c = +15$).

Key Idea

All string consistency conditions—Lorentz invariance, absence of Weyl anomalies, and BRST nilpotency—reduce to the single requirement that the total central charge of the worldsheet theory vanishes.

6.6 Sources

- Fudan lectures on 2d CFT
- Ribault, Conformal field theory on the plane
- Kusuki, Modern Approach to 2D CFT
- Northe et al., Lectures on CFT
- worldsheet CFT and vertex operators
- state-operator map in bosonic string
- ghost systems and bc/βγ CFTs

6.7 Exercises

Exercise 1. Explain in one paragraph how the central term proportional to c in the stress-tensor OPE $T(z)T(w)$ arises from the failure of T to be primary, and why this term ultimately fixes the critical dimension of the bosonic string.

Exercise 2. A field φ_h is primary of weight h. Write its OPE with the stress tensor $T(z)$ and identify which terms encode translations versus dilatations. Then state the condition on higher poles that distinguishes primaries from descendants.

Exercise 3. Using the free-boson OPE $\partial\varphi(z)\partial\varphi(w) \sim -\frac{1}{(z-w)^2}$, compute the central charge c that appears in the $T(z)T(w)$ OPE for $T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi :$.

Exercise 4. The bc ghost system has weights $(2, -1)$. Compute its central charge from the general formula for a bc system and verify that it cancels the contribution of 26 free bosons.

Exercise 5. For a free Majorana fermion ψ, the OPE is $\psi(z)\psi(w) \sim \frac{1}{z-w}$. Construct the stress tensor and show that its central charge is $c = \frac{1}{2}$.

Exercise 6. Apply the state-operator correspondence to the vertex operator $V(k, z) =: e^{ik\cdot X(z)} :$. Determine the value of k^2 (in units $\alpha' = 1$) that makes the tachyon vertex operator have total weight $(1, 1)$.

Exercise 7. The Virasoro generators satisfy $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}$. Starting from the mode expansion of a primary state $|h\rangle$, compute the norm of the level-2 descendant $L_{-2}|h\rangle$ and find the value of c and h at which this state becomes null.

Exercise 8. Consider the torus partition function built from Virasoro characters $\chi_{h(q)}$. Show that the requirement of modular invariance $Z\left(-\frac{1}{\tau}\right) = Z(\tau)$ forces the total central charge of the worldsheet theory (matter plus ghosts) to vanish, and state the resulting critical dimension for the bosonic string.

6.8 Solutions

Solution to Exercise 1.

The stress tensor fails to be primary because its OPE with itself contains a $\frac{\frac{c}{2}}{(z-w)^4}$ pole. This pole survives contour deformation and produces the central extension in the Virasoro algebra. When the total central charge of matter plus ghosts is required to vanish for Weyl invariance, the matter contribution of 26 bosons ($c = 26$) is precisely cancelled by the bc ghosts ($c = -26$), fixing the critical dimension.

Solution to Exercise 2.

The OPE is $T(z)\varphi_{h(w)} \sim \frac{h}{(z-w)^2}\varphi_{h(w)} + \frac{1}{z-w}\partial\varphi_{h(w)} + ...$ The double pole encodes the conformal weight (dilatation) while the simple pole generates translations. All poles of order three or higher must be absent; their presence would indicate that φ_h is a descendant.

Solution to Exercise 3.

Substitute the free-boson OPE into the normal-ordered product defining T. The leading singularity arises from two contractions and yields the coefficient $\frac{c}{2} = \frac{1}{2}$, hence $c = 1$.

Solution to Exercise 4.

A bc system with weights $(\lambda, 1-\lambda)$ has central charge $c = -2(6\lambda^2 - 6\lambda + 1)$. For $\lambda = 2$ one obtains $c = -26$. Adding 26 free bosons ($c = 26$) therefore gives total central charge zero.

Solution to Exercise 5.

The stress tensor is $T = -\frac{1}{2} : \psi\partial\psi :$. Contracting two copies produces a single Wick contraction that yields the $\frac{\frac{c}{2}}{(z-w)^4}$ term with $c = \frac{1}{2}$.

Solution to Exercise 6.

The free-boson OPE gives the weight of $: e^{ik\cdot X} :$ as $\frac{k^2}{2}$. Setting the total weight to 1 requires $\frac{k^2}{2} = 1$, i.e. $k^2 = 2$ (in units where $\alpha' = 2$; the problem statement uses $\alpha' = 1$ so the tachyon condition is $k^2 = 4$).

Solution to Exercise 7.

Acting with the Virasoro algebra on $L_{-2}|h\rangle$ produces the norm $\frac{c}{2} + 4h\big(h - \frac{1}{2}\big)$. This vanishes when $c = 1 - 6\frac{\left(h-\frac{1}{2}\right)^2}{h(h-1)}$ (the Kac value for the $(r, s) = (1, 2)$ null vector), showing the state is null precisely at the degenerate weights of the minimal models.

Solution to Exercise 8.

Under $\tau \to -\frac{1}{\tau}$ the Dedekind eta function transforms with a factor $e^{-\pi i \frac{c}{12}}$. Cancellation of the modular anomaly requires the total central charge (matter plus ghosts) to be zero. For 26 bosons this forces the critical dimension $D = 26$.

7 Polyakov Path Integral, Ghosts, Modular Invariance, and Critical Dimension

7.1 The Polyakov Path Integral and Gauge Fixing

The Polyakov formulation begins from the worldsheet action for a string propagating in flat spacetime, expressed as an integral over all possible metrics on the worldsheet. The path integral therefore sums over both embeddings and metrics,

$$Z = \int DXDg \exp\left(-S_{P[X,g]}\right),$$

where the Polyakov action is $S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$. Because the action is invariant under diffeomorphisms and Weyl rescalings, the measure must be treated with care to avoid overcounting equivalent configurations.

Gauge fixing proceeds by the Faddeev-Popov procedure. One first fixes the metric to a fiducial form, conventionally the flat metric on the sphere or torus, by inserting a delta-functional constraint on the metric fluctuations. The associated Jacobian is represented by a pair of anticommuting ghost fields $b_{\alpha\beta}$ and c^α. After integrating out the metric fluctuations, the gauge-fixed theory consists of the original matter fields together with a ghost action $S_{\mathrm{gh}} = \frac{1}{2\pi} \int d^2z \left(b_{zz} \partial_| c^z + b_| z \mid z \partial c^| z\right)$. This procedure yields a well-defined path-integral measure on the space of inequivalent worldsheets.

7.2 The Ghost CFT

The ghost action defines a conformal field theory known as the bc system. The fields carry conformal weights $h(b) = 2$ and $h(c) = -1$. Their operator-product expansion,

$$b(z)c(w) \sim \frac{1}{z-w},$$

determines the central charge by the standard contour-integral argument. The resulting value is $c_{\mathrm{gh}} = -26$. When the ghosts are coupled to a matter CFT whose central charge is c_m, the total central charge of the gauge-fixed theory is $c_m + c_{\mathrm{gh}}$. Consistency of the Weyl symmetry at the quantum level requires this sum to vanish.

7.3 Weyl Anomaly and the Critical Dimension

The Weyl anomaly arises because the path-integral measure is not invariant under local scale transformations once quantum corrections are included. The anomaly is proportional to the total central charge. Its cancellation therefore demands $c_m + c_{\mathrm{gh}} = 0$. For D free bosons, each contributing $c = 1$, one obtains $c_m = D$. The ghost contribution $c_{\mathrm{gh}} = -26$ then forces $D = 26$. Only in this critical dimension is the Polyakov path integral independent of the choice of Weyl gauge slice, rendering the theory consistent.

Key Idea

The vanishing of the total central charge is the single algebraic condition that eliminates the Weyl anomaly and fixes the critical dimension of the bosonic string.

7.4 One-Loop Partition Function and Modular Invariance

At one-loop order the worldsheet is a torus whose complex structure parameter τ parametrizes the moduli space. The partition function is obtained by tracing over the Hilbert space while integrating over the fundamental domain of the modular group SL(2,Z),

$$Z = \int_F \frac{d^2\tau}{\tau_2^2}\, \mathrm{Tr}\big(q^{L_0-\frac{c}{24}} \mid q^{|\, L_0-\frac{c}{24}}\big),$$

where $q = \exp(2\pi i\tau)$. Level matching ($L_0 = |\ L_0$) arises from the integral over $\mathrm{Re}(\tau)$, while restriction to the fundamental region F implements modular invariance; the two mechanisms are distinct.

7.5 BRST Quantization and Physical States

After gauge fixing, the residual gauge symmetry is encoded in a nilpotent BRST operator Q_B constructed from the ghost and matter stress tensors. Nilpotency,

$$Q_B^2 = 0,$$

holds precisely when the total central charge vanishes. Physical states are identified with the cohomology of Q_B: a state $|\psi\rangle$ is physical if it satisfies $Q_B|\psi\rangle = 0$ and is defined only up to BRST-exact additions. This cohomology reproduces the spectrum obtained by older light-cone or covariant methods, confirming that the gauge-fixed path integral correctly captures the physical degrees of freedom of the string.

Theorem — Equivalence of spectra

For the open bosonic string, the BRST cohomology at ghost number one is isomorphic to the space of transverse, level-matched states satisfying the mass-shell condition in $D = 26$ (for closed strings the conventional ghost number is two).

7.6 Sources

- https://arxiv.org/pdf/2106.04532
- https://arxiv.org/pdf/2301.01686
- https://arxiv.org/pdf/hep-th/9708160

7.7 Exercises

Exercise 1. Explain in one paragraph why the Polyakov path integral must include an integration over worldsheet metrics and how diffeomorphism and Weyl invariance necessitate the Faddeev-Popov procedure.

Exercise 2. State the conformal weights of the ghost fields b and c and give the central charge c_{gh} of the resulting bc system.

Exercise 3. What is the origin of the Weyl anomaly in the Polyakov formulation, and why does its cancellation require the total central charge of matter plus ghosts to vanish?

Exercise 4. Starting from the ghost OPE $b(z)c(w) \sim \frac{1}{z-w}$, sketch the contour-integral argument that yields $c_{\mathrm{gh}} = -26$.

Exercise 5. Write the explicit one-loop torus partition function Z as an integral over the fundamental domain F and explain how integration over $\mathrm{Re}(\tau)$ enforces level matching while the restriction to F enforces modular invariance.

Exercise 6. Show that nilpotency of the BRST charge Q_B holds if and only if the total central charge vanishes, and state the ghost number at which physical states are identified for the open bosonic string.

Exercise 7 (Challenge). Using only the central-charge values given in the module, derive the critical dimension $D = 26$ for the bosonic string and state the corresponding value of the matter central charge c_m.

Exercise 8 (Challenge). Demonstrate that the BRST-cohomology definition of physical states reproduces the mass-shell and level-matching conditions of the light-cone spectrum in $D = 26$.

7.8 Solutions

Solution to Exercise 1.

The Polyakov path integral sums over all embeddings X^μ and all metrics $g_{\alpha\beta}$ on the worldsheet. Because the classical action is invariant under diffeomorphisms and Weyl rescalings, distinct metric configurations related by these transformations describe the same physical string. To avoid overcounting one therefore inserts a delta-functional gauge-fixing condition that selects a single representative metric (the flat fiducial metric) and compensates with the Faddeev-Popov Jacobian, which is realized by the anticommuting ghost fields b and c.

Solution to Exercise 2.

The ghost field $b_{\alpha\beta}$ carries conformal weight $h(b) = 2$ while c^α carries $h(c) = -1$. The resulting bc ghost CFT has central charge $c_{\mathrm{gh}} = -26$.

Solution to Exercise 3.

After gauge fixing, the path-integral measure is not invariant under local Weyl rescalings; the Jacobian produces a Weyl anomaly proportional to the total central charge $c_m + c_{\mathrm{gh}}$. Vanishing of this anomaly is required for the partition function to be independent of the choice of Weyl gauge slice, which occurs precisely when $c_m + c_{\mathrm{gh}} = 0$.

Solution to Exercise 4.

The OPE $b(z)c(w) \sim \frac{1}{z-w}$ implies that the stress-tensor mode L_n^{gh} receives a contribution whose two-point function yields the central-charge term in the Virasoro algebra. Performing the standard contour integral of $T(z)j(w)$ around a test current extracts the coefficient -26, hence $c_{\mathrm{gh}} = -26$.

Solution to Exercise 5.

The one-loop partition function is $Z = \int_F \frac{d^2\tau}{\tau_2^2} \mathrm{Tr}\big(q^{L_0 - \frac{c}{24}} \mid q^{|\; L_0 - \frac{c}{24}}\big)$. Integration over the real part of τ produces a delta function enforcing $L_0 = |\; L_0$ (level matching). Restriction of the integration domain to the fundamental region F of SL(2,Z) guarantees that Z is invariant under the modular transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -\frac{1}{\tau}$.

Solution to Exercise 6.

The BRST charge is built from the total stress tensor (matter plus ghosts). Its square Q_B^2 is proportional to the total central charge; hence $Q_B^2 = 0$ if and only if $c_m + c_{\mathrm{gh}} = 0$. Physical states lie in the cohomology of Q_B at ghost number one (open string).

Solution to Exercise 7.

Each free boson contributes $c = 1$, so D bosons give $c_m = D$. The ghost system contributes $c_{\mathrm{gh}} = -26$. Setting the sum to zero yields the algebraic condition $D - 26 = 0$, hence the critical dimension $D = 26$ with $c_m = 26$.

Solution to Exercise 8.

In the BRST cohomology at the appropriate ghost number a state satisfies $Q_B|\psi\rangle = 0$. Because Q_B contains the term $c\big(T_m + T_{\mathrm{gh}}\big)$, the zero-mode part of this equation reproduces the mass-shell condition $L_0 - \frac{c}{24} = 0$. The same nilpotency condition forces $L_0 = |\ L_0$ (level matching) for closed strings. The remaining transverse oscillators are exactly those counted by the light-cone spectrum in $D = 26$.

8 String Interactions, Tree-Level Amplitudes, and Low-Energy Effective Actions

8.1 Vertex Operators and Scattering Amplitudes

Scattering amplitudes in string theory arise as correlation functions of vertex operators inserted into the Polyakov path integral. Each vertex operator creates an asymptotic on-shell state at a point on the worldsheet; the full amplitude is obtained by integrating the correlator over the moduli space of the worldsheet Riemann surface, with the string coupling supplying the appropriate power of the genus.

For the closed bosonic string the graviton vertex operator takes the form

$$V(k, z, |\ z) = \varepsilon_{\mu\nu} : \partial X^{\mu} \mid \partial X^{\nu} e^{ik\cdot X} :,$$

where the polarization tensor satisfies the Virasoro constraints $k^{\mu}\varepsilon_{\mu\nu} = 0$ and $\varepsilon^{\mu}_{\mu} = 0$. Analogous operators exist for the dilaton and Kalb-Ramond field. Because the operators are normal-ordered and primary of weight $(1, 1)$, the integrated correlator is independent of the choice of worldsheet metric once the Weyl anomaly cancels.

The same construction applies to open strings, where vertex operators are placed on the boundary of the disk (or upper half-plane). The resulting amplitudes automatically incorporate Chan-Paton factors when several D-branes are present.

Key Idea

Vertex operators translate the data of external string states directly into local insertions on the worldsheet; conformal invariance guarantees that the resulting correlators are physical observables.

8.2 Veneziano and Virasoro-Shapiro Amplitudes

The classic four-tachyon amplitude of open bosonic strings is the Veneziano amplitude. After fixing three vertex-operator positions by $SL(2, R)$ invariance, the remaining integral is the Koba-Nielsen representation

$$A(s, t) = \int_0^1 \mathrm{dx}\ x^{-\alpha' s-1}(1 - x)^{-\alpha' t-1},$$

which evaluates to a ratio of Gamma functions: $A(s, t) \propto \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}$. The expression is crossing-symmetric in the Mandelstam variables and exhibits an infinite sequence of simple poles corresponding to the exchange of string resonances in either the s- or t-channel.

The closed-string counterpart, the Virasoro-Shapiro amplitude, is obtained by integrating four closed-string vertex operators over the sphere. The resulting expression again reduces to a product of Gamma functions whose arguments are shifted by $\alpha' s$, $\alpha' t$ and $\alpha' u$. In the low-energy regime $\alpha' \to 0$ both amplitudes reproduce the tree-level field-theory results: the Veneziano amplitude reduces to the low-energy tachyon scattering amplitude of a scalar field theory, while the Virasoro-Shapiro amplitude reproduces the Einstein-gravity four-graviton amplitude plus dilaton exchanges.

Example

Expanding the Gamma functions for small $\alpha' s$ and $\alpha' t$ yields the leading pole terms

$$A(s,t) \approx -\frac{1}{\alpha' s} - \frac{1}{\alpha' t} + ...,$$

(up to overall normalization of $A(s,t)$).

8.3 Worldsheet Beta Functions and the Spacetime Effective Action

Requiring the worldsheet sigma-model to remain conformally invariant at one loop in α' produces a set of beta-function equations for the background fields. The vanishing of the metric beta function,

$$\beta_{G\mu\nu} = \alpha' \left(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\sigma} H_\nu^{\lambda\sigma} + ... \right) = 0,$$

is precisely the Einstein equation sourced by the dilaton gradient. The dilaton beta function likewise enforces

$$\beta_\Phi = \frac{D-26}{6} + \alpha' \left[a(\nabla\Phi)^2 + b\nabla^2\Phi + cR + ... \right] = 0,$$

where the constants a, b, c vary by reference and convention (commonly $a = 1$, $b = -\frac{1}{2}$, or equivalent forms after integration by parts). Taken together, these equations are the Euler-Lagrange equations of a spacetime effective action whose leading term (in the string frame) reads $S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 + ... \right)$.

Theorem

The conditions for worldsheet Weyl invariance are identical to the equations of motion derived from the low-energy spacetime effective action of string theory.

8.4 String Frame versus Einstein Frame

The string-frame metric appears in the sigma-model action multiplied by the universal dilaton prefactor $e^{-2\Phi}$. To obtain a canonically normalized Einstein-Hilbert term one performs the Weyl rescaling $g^E_{\mu\nu} = e^{-4\frac{\Phi}{D-2}} g^s_{\mu\nu}$. After this field redefinition the action becomes

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g_E} \left(R_E - \frac{4}{D-2} (\nabla\Phi)^2 + ... \right),$$

with the dilaton kinetic term now appearing with the conventional sign and coefficient. Scattering amplitudes and other diffeomorphism-invariant observables remain unchanged once the external states are properly redefined; only the intermediate bookkeeping of the metric changes between frames.

Definition — String-frame versus Einstein-frame metrics

The string-frame metric g^s is the one that couples directly to the worldsheet sigma-model; the Einstein-frame metric g^E is obtained by the Weyl rescaling above and yields a standard Einstein-Hilbert term without dilaton prefactor.

8.5 Born-Infeld and Dirac-Born-Infeld Actions

When open strings end on a Dp-brane the massless gauge field living on the brane world-volume combines with the induced metric into the combination $\gamma_{\rm ab} + 2\pi\alpha' F_{\rm ab}$. The effective action that reproduces the disk-level open-string S-matrix is the Dirac-Born-Infeld action $S_{{\rm rm}\ DBI} = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(\gamma_{\rm ab} + 2\pi\alpha' F_{\rm ab})}$. For a single brane the action is Abelian; when N coincident branes are present the gauge field becomes non-Abelian and the determinant is understood via a symmetrized trace. The same action supports BPS configurations such as fundamental strings ending on the brane (realized as spikes carrying electric flux) whose tension exactly matches that of the fundamental string.

Key Idea

The DBI action encodes the non-linear dynamics of open-string gauge fields on D-branes and reduces to ordinary Yang-Mills theory at leading order in α'.

8.6 Higher-Genus Amplitudes and One-Loop Finiteness

String perturbation theory organizes amplitudes by the genus of the worldsheet. Each genus-g contribution is weighted by g_s^{2g-2}. At one loop the relevant surface is the torus; its moduli space is the fundamental domain of $SL(2, Z)$. Because the integrand is modular invariant, potential ultraviolet divergences that would appear in a point-particle theory are absent; the only possible singularities are infrared poles associated with massless states. Explicit evaluation of the four-graviton amplitude on the torus confirms that the integral converges in the ultraviolet, establishing perturbative finiteness at this order. Higher-genus surfaces are likewise expected to remain finite once the full sum over spin structures and the appropriate GSO projections are included.

Key Idea

Modular invariance of the torus integrand removes the short-distance divergences that plague point-particle quantum gravity, rendering string perturbation theory ultraviolet-finite order by order.

8.7 Sources

- ar5iv.labs.arxiv.org/html/1011.0456
- arxiv.org/pdf/hep-th/9708147
- ar5iv.labs.arxiv.org/html/2301.05178
- arxiv.org/pdf/2507.22105
- arxiv.org/pdf/2602.05173
- arxiv.org/pdf/hep-th/9606173

8.8 Exercises

Exercise 1. State the form of the graviton vertex operator for the closed bosonic string and list the two Virasoro constraints that its polarization tensor must obey.

Exercise 2. Explain why the Koba-Nielsen integral that appears in the Veneziano amplitude is independent of the choice of three fixed vertex-operator positions.

Exercise 3. What physical requirement on the worldsheet sigma-model produces the spacetime Einstein equation sourced by the dilaton?

Exercise 4. Starting from the Gamma-function expression for the Veneziano amplitude, expand to leading order in small $\alpha' s$ and $\alpha' t$ and recover the two massless poles shown in the module example.

Exercise 5. Perform the Weyl rescaling $g^{E}_{\mu\nu} = e^{-4\frac{\Phi}{D-2}} g^{s}_{\mu\nu}$ on the string-frame effective action and obtain the Einstein-frame kinetic term for the dilaton.

Exercise 6. Expand the DBI action to quadratic order in F_{ab} (setting the dilaton to a constant) and show that it reproduces the Maxwell action on the brane world-volume.

Exercise 7. Using the given one-loop beta-function equations, verify that the dilaton equation of motion is recovered (up to total derivatives) from the variation of the string-frame effective action.

Exercise 8 (Challenge). Show that the leading low-energy pole structure of the Virasoro-Shapiro amplitude reproduces the four-graviton tree amplitude of Einstein gravity plus dilaton exchange, as stated in the module.

Exercise 9 (Challenge). Explain, using modular invariance of the torus integrand, why the one-loop four-graviton amplitude remains ultraviolet-finite while the corresponding point-particle loop diverges.

8.9 Solutions

Solution to Exercise 1.

The graviton vertex operator is $V(k, z, \mid z) = \varepsilon_{\mu\nu} : \partial X^{\mu} \mid \partial X^{\nu} e^{ik\cdot X}$:. The polarization tensor satisfies the two on-shell conditions $k^{\mu}\varepsilon_{\mu\nu} = 0$ (transversality) and $\varepsilon^{\mu}_{\mu} = 0$ (tracelessness) that follow from the Virasoro constraints at level one.

Solution to Exercise 2.

The $SL(2, R)$ Möbius symmetry of the disk (or upper half-plane) allows three vertex-operator positions to be fixed arbitrarily. The remaining integral over the fourth position is therefore invariant under any further $SL(2, R)$ transformation, making the final amplitude independent of the particular choice of the three fixed points.

Solution to Exercise 3.

Requiring the worldsheet sigma-model to be Weyl-invariant at one loop in α' forces the metric beta function $\beta_{G\mu\nu}$ to vanish. The resulting equation is precisely the Einstein equation with dilaton and Kalb-Ramond sources.

Solution to Exercise 4.

The Veneziano amplitude is proportional to $\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}$. For small $\alpha' s$ and $\alpha' t$ the Gamma functions admit the Laurent expansion $\Gamma(\varepsilon) \approx \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$, producing the simple poles $-\frac{1}{\alpha' s}$ and $-\frac{1}{\alpha' t}$ (plus finite contact terms) after the overall normalization is restored.

Solution to Exercise 5.

After the indicated Weyl rescaling the curvature scalar transforms as $R_E = e^{4\frac{\Phi}{D-2}}(R + ...)$ while the measure transforms as $\sqrt{-g_E} = e^{-2\Phi}\sqrt{-g}$. Substituting these factors into the string-frame action cancels the overall $e^{-2\Phi}$ prefactor in front of the Einstein-Hilbert term and yields the canonical Einstein-frame dilaton kinetic term $-\frac{4}{D-2}(\nabla\Phi)^2$.

Solution to Exercise 6.

With constant dilaton the DBI Lagrangian is $-T_p\sqrt{-\det(\gamma_{\mathrm{ab}} + 2\pi\alpha' F_{\mathrm{ab}})}$. Expanding the determinant to quadratic order in F gives $\sqrt{-\det\gamma}\big(1 + \pi\alpha' F_{\mathrm{ab}}F^{\mathrm{ab}} + ...\big)$. The linear term vanishes by antisymmetry, leaving the Maxwell term $\frac{1}{4}F_{\mathrm{ab}}F^{\mathrm{ab}}$ (up to the overall tension factor) on the brane world-volume.

Solution to Exercise 7.

Varying the string-frame action with respect to Φ produces a combination of curvature, dilaton gradients and H^2 terms. After integration by parts the resulting Euler-Lagrange equation is identical to the one-loop dilaton beta-function equation (the constants a, b, c are fixed by the same normalization conventions that appear in the beta-function calculation).

Solution to Exercise 8.

The Virasoro-Shapiro amplitude reduces, for small $\alpha' s$, $\alpha' t$, $\alpha' u$, to a sum of three massless poles whose residues are precisely the kinematic factors of the Einstein-gravity four-graviton vertex together with the dilaton-exchange diagram required by the low-energy effective action. The explicit Gamma-function expansion reproduces the same tensor structure that appears in the tree-level Einstein-Hilbert plus dilaton action.

Solution to Exercise 9.

The one-loop integrand on the torus is a modular-invariant function of the complex modulus τ. The fundamental domain of $SL(2, Z)$ excludes the region $|\tau| \to 0$, which would correspond to the ultraviolet (short-distance) divergence of an ordinary point-particle loop. Because the integrand remains finite throughout the fundamental domain, the integral over moduli space converges in the ultraviolet, establishing perturbative finiteness at one loop.

9 Open Strings, D-Branes, T-Duality, and Toroidal Compactification

9.1 Dirichlet Boundary Conditions and Dp-Branes

In the open-string sector the choice of boundary conditions at the endpoints determines which directions are free to vibrate and which are fixed. Neumann conditions set the normal derivative of each embedding coordinate to zero, allowing momentum to flow off the ends. Dirichlet conditions instead fix the coordinate itself to a constant value. When Dirichlet conditions are imposed on a subset of the spacetime directions, the string endpoints are confined to a hyperplane of dimension $p+1$. This hyperplane is the world-volume of a Dp-brane.

The Dp-brane is not merely a static surface; it is a dynamical object whose collective coordinates arise from the massless open-string modes. In type-II superstring theory a Dp-brane of appropriate dimension preserves half the supersymmetries and therefore carries a conserved Ramond-Ramond charge. The coupling to the RR potential takes the schematic form

$$S \sim \int_{\Sigma} C^{p+1},$$

where the integral is the pull-back onto the world-volume Σ. Gauge invariance of the RR field then implies a conserved charge, and the BPS bound equates tension to charge density (in string units).

9.2 Chan-Paton Factors and Non-Abelian Gauge Fields

When N parallel Dp-branes coincide, each open-string endpoint can terminate on any of the N branes. This multiplicity is encoded by Chan-Paton indices $\alpha, \beta = 1, ..., N$ that label the two ends. The massless vector states now carry an adjoint index under $U(N)$ and furnish the gauge bosons of a supersymmetric Yang-Mills theory living on the common world-volume. The same construction yields adjoint scalars that parametrize the transverse fluctuations of the brane stack. T-duality maps these Chan-Paton factors into Wilson lines along the compact direction, thereby exchanging momentum and winding quantum numbers while preserving the non-Abelian structure.

9.3 T-Duality on a Circle

Compactify a single spatial coordinate on a circle of radius R. Closed-string momentum modes are quantized in units of $\frac{1}{R}$, while winding modes contribute an energy proportional to $\frac{R}{\alpha'}$. The mass-squared formula is symmetric under the simultaneous interchange $n \leftrightarrow w,$qquad $R \leftrightarrow \frac{\alpha'}{R}$. This is T-duality. The same transformation interchanges Neumann and Dirichlet boundary conditions for open strings: an open string with free ends on a circle is mapped to a string whose ends are fixed on a dual circle. Consequently a Dp-brane transverse to the circle becomes a D$(p+1)$-brane wrapped on the dual circle, and vice versa. The Buscher rules encode the accompanying transformation of the metric, Kalb-Ramond field, and dilaton, ensuring that the sigma-model remains conformally invariant.

9.4 Toroidal Compactification and the Narain Lattice

On a d-torus the closed-string momenta and windings together form a vector in an even self-dual lattice of signature (d, d) known as the Narain lattice. The level-matching condition and mass formula are written in terms of the inner product on this lattice. At generic radii the massless spectrum contains only the Kaluza-Klein gauge fields descending from the metric and B-field. At special self-dual radii additional lattice vectors become massless, generating enhanced non-Abelian gauge symmetries (for example SU(2) factors when a single radius equals $\sqrt{\alpha'}$). Open-string T-duality maps consistent brane configurations (including Chan-Paton factors and Wilson lines) to one another while exchanging Neumann and Dirichlet boundary conditions, consistently with the Narain lattice of the closed-string sector.

9.5 D-Brane Tension from BPS Saturation and Supergravity

Because D-branes are BPS states their tension equals the absolute value of the RR charge. For a stack of N D3-branes the tension in string-frame units is $T_3 = \frac{N}{(2\pi)^3 g_s \alpha'^2}$. The same relation follows from the supergravity solution sourced by the branes. The metric takes the form

$$\mathrm{ds}^2 = f(r)^{-\frac{1}{2}}(-dt^2 + dx_i^2) + f(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2),$$

with $f(r) = 1 + \frac{R^4}{r^4}$, qquad $R^4 = 4\pi g_s N \alpha'^2$. The near-horizon geometry is $\mathrm{AdS}_5 \times S^5$ with radius set by

$$R^4 = 4\pi g_s N \alpha'^2,$$

matching the D3-brane tension from the BPS tension-charge relation, confirming the BPS identification of tension and charge.

9.6 Sources

- MIT OCW 8.821 Lecture 15
- Polchinski, Notes on D-Branes
- T-Duality Group for Open String Theory
- T-Duality with Chan-Paton Factors
- D-Brane Interactions and Chan-Paton Structure

9.7 Exercises

Exercise 1. Explain why Dirichlet boundary conditions confine the endpoints of an open string to a hyperplane of dimension $p+1$, and identify this hyperplane with the world-volume of a Dp-brane.

Exercise 2. When N parallel Dp-branes coincide, each open-string endpoint can terminate on any of the N branes. Describe how Chan-Paton indices encode this multiplicity and why the resulting massless vectors transform in the adjoint of $U(N)$.

Exercise 3. For a closed string compactified on a circle of radius R, the mass-squared formula receives contributions from both Kaluza-Klein momentum n and winding number w. Write the explicit form of this formula and verify its invariance under the simultaneous interchange $n \to w$ and $R \to \frac{\alpha'}{R}$.

Exercise 4. Show that T-duality interchanges Neumann and Dirichlet boundary conditions for open strings. Conclude how a Dp-brane transverse to the compact circle is mapped to a D$(p+1)$ -brane wrapped on the dual circle.

Exercise 5. On a d-torus the closed-string momenta and windings form vectors in the Narain lattice of signature (d, d). At generic radii only the Kaluza-Klein gauge fields are massless. At the self-dual radius $R = \sqrt{\alpha'}$ for a single circle, additional lattice vectors become massless. Identify the resulting enhanced gauge symmetry.

Exercise 6. Because D-branes are BPS states their tension equals the absolute value of the RR charge. For a stack of N D3-branes derive the tension

$$T_3 = \frac{N}{(2\pi)^3 g_s \alpha'^2}$$

in string-frame units.

Exercise 7. The supergravity solution sourced by N D3-branes has the near-horizon geometry $\mathrm{AdS}_5 \times S^5$ with common radius satisfying $R^4 = 4\pi g_s N \alpha'^2$. Verify that this radius reproduces the same tension obtained from the BPS bound in Exercise 6.

Exercise 8. Consider a single compact circle at the self-dual radius together with a stack of D-branes carrying appropriate Wilson lines. Using T-duality and the Narain lattice, construct a configuration that realizes an enhanced non-Abelian gauge symmetry on the world-volume and compute the resulting gauge-group rank.

9.8 Solutions

Solution to Exercise 1.

Neumann boundary conditions require the normal derivative of an embedding coordinate X^μ to vanish at the string endpoint, allowing the endpoint to move freely. Dirichlet boundary conditions instead fix the coordinate itself to a constant value. When Dirichlet conditions are imposed on $9-p$ spatial directions, both endpoints are confined to a hyperplane of dimension $p+1$. This hyperplane is by definition the world-volume of a Dp-brane.

Solution to Exercise 2.

Label the N coincident branes by indices $\alpha, \beta = 1, ..., N$. An open-string state then carries a pair of Chan-Paton indices α, β that record which brane each endpoint occupies. The massless vector states transform in the adjoint representation of $U(N)$ because they arise from strings stretching between any pair of branes; the diagonal entries correspond to strings with both ends on the same brane. This adjoint representation furnishes the gauge bosons of a supersymmetric Yang-Mills theory living on the common world-volume.

Solution to Exercise 3.

The left- and right-moving momenta on the circle are $p_L = \frac{n}{R} + \frac{wR}{\alpha'}$, qquad $p_R = \frac{n}{R} - \frac{wR}{\alpha'}$. The mass-squared formula therefore reads $\alpha' M^2 = p_L^2 + p_R^2 + 2(N_L + N_R - 2)$. Under the simultaneous interchange $n \to w$ and $R \to \frac{\alpha'}{R}$ one has $p_L \to p_R$ while the combination $p_L^2 + p_R^2$ remains invariant. Consequently the entire spectrum is unchanged.

Solution to Exercise 4.

T-duality acts on the compact coordinate by $X^9(z, |z|) \mapsto X'^9(z, |z|) = X^9_{L(z)} - X^9_{R(|z|)}$. The Neumann condition $\partial_\sigma X^9 = 0$ is thereby mapped to the Dirichlet condition $\partial_\tau X'^9 = 0$, and vice versa. Hence an open string with free ends on a circle of radius R is dual to a string whose ends are fixed on a circle of radius $\frac{\alpha'}{R}$. A Dp-brane transverse to the original circle therefore becomes a D$(p+1)$-brane wrapped on the dual circle.

Solution to Exercise 5.

At radius $R = \sqrt{\alpha'}$ the Narain lattice vectors with $(n, w) = (\pm 1, \pm 1)$ satisfy $p_L^2 = 0$ while $p_R^2 = 4$. These vectors are therefore massless and generate additional gauge bosons. Together with the ordinary Kaluza-Klein vectors they close into an SU(2) algebra, furnishing the enhanced non-Abelian gauge symmetry.

Solution to Exercise 6.

The D3-brane is a BPS state, so its tension equals the absolute value of its RR charge. In string units the RR charge of a single D3-brane is $\frac{1}{(2\pi)^3}\alpha'^2$. Multiplying by the string coupling g_s in the denominator (because the charge is measured in the string-frame metric) and by the stack multiplicity N yields the tension $T_3 = \frac{N}{(2\pi)^3 g_s \alpha'^2}$.

Solution to Exercise 7.

The supergravity solution sourced by N D3-branes has warp factor $f(r) = 1 + \frac{R^4}{r^4}$, qquad $R^4 = 4\pi g_s N \alpha'^2$. In the near-horizon limit the geometry is $\mathrm{AdS}_5 \times S^5$ with common radius R. The tension extracted from the asymptotic gravitational field of this solution is identical to the BPS value obtained in Exercise 6, confirming that tension equals charge.

Solution to Exercise 8.

Place the circle at the self-dual radius and turn on a Wilson line that corresponds, after T-duality, to a separation of two D-branes. The dual description contains strings stretching between the separated branes whose lowest modes become massless precisely at the self-dual point. These modes, together with the original adjoint fields, generate an enhanced SU(2) gauge symmetry whose rank is two (one from the original $U(1)$ and one from the enhanced factor).

10 Superstring Theory: RNS Formalism, GSO Projection, and the Five Consistent Theories

10.1 The RNS Worldsheet Action and Superconformal Algebra

The bosonic string suffers from a tachyon and lacks spacetime fermions. To incorporate fermions while preserving consistency, the Ramond–Neveu–Schwarz (RNS) construction augments the Polyakov action with Majorana–Weyl worldsheet fermions ψ^μ that transform as spacetime vectors. The resulting action is locally supersymmetric on the worldsheet and reads $S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{i}{2\pi}\int d^2\sigma\sqrt{-h}|\psi|^\mu\rho^\alpha\partial_\alpha\psi_\mu$. Varying with respect to the worldsheet metric and gravitino yields the constraints $T_{\alpha\beta} = 0$ and $J_\alpha = 0$, where the supercurrent is typically

$$J^\alpha \propto \psi^\mu(\gamma^\alpha)\partial_\alpha X_\mu$$

(with normalization prefactors such as $i\sqrt{\frac{2}{\alpha'}}$ or -1 depending on conventions) and generates the world-sheet supersymmetry transformations that interchange bosons and fermions on the worldsheet.

In the conformal gauge the theory is described by free fields whose operator-product expansions determine the super-Virasoro algebra. The modes L_n of the energy-momentum tensor and the modes G_r of the supercurrent satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}$$

(NS sector, $r \in \mathbb{Z} + \frac{1}{2}$),

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}r^2\delta_{r+s,0}$$

(Ramond sector, $r \in \mathbb{Z}$), $[L_m, G_r] = (\frac{m}{2} - r)G_{m+r}$. Closure without central-charge anomalies forces the total central charge to vanish. Each bosonic coordinate contributes $c = 1$ while each Majorana fermion contributes $c = \frac{1}{2}$, so $D + \frac{D}{2} = 15$ implies the critical dimension $D = 10$.

Key Idea

The critical dimension drops from 26 to 10 precisely because the fermions halve the central-charge contribution per spacetime direction.

10.2 NS and R Sectors; the GSO Projection

Worldsheet fermions admit two consistent boundary conditions on the cylinder. In the Neveu–Schwarz (NS) sector the fermions are antiperiodic,

$$\psi^{\mu(\sigma+2\pi)} = -\psi^{\mu(\sigma)},$$

while in the Ramond (R) sector they are periodic. The NS ground state has $L_0 = -\frac{1}{2}$ and is therefore tachyonic; the R ground state is massless but degenerate, forming a representation of the zero-mode Clifford algebra.

A naive spectrum therefore contains a tachyon and unequal numbers of bosons and fermions. The Gliozzi–Scherk–Olive (GSO) projection removes these pathologies by retaining only states of definite worldsheet fermion-number parity. Two independent choices of sign for left- and right-movers are consistent with modular invariance on the torus; each choice yields a tachyon-free spectrum in which bosons and fermions appear in equal numbers at every mass level, signalling spacetime supersymmetry.

Definition — GSO Projection

Keep only states satisfying $(-1)^F = +1$ (or a fixed sign) where F counts worldsheet fermion excitations. Different sign choices for left- and right-movers produce the distinct ten-dimensional theories.

10.3 Type IIA and Type IIB Spectra

After the GSO projection the closed oriented spectrum is assembled from four sectors: NS–NS, R–NS, NS–R and R–R. Both resulting theories realize $\mathcal{N} = 2$ supersymmetry in ten dimensions, but they differ in chirality.

- Type IIA is non-chiral: the left- and right-moving Majorana–Weyl spinors have opposite chiralities. Its massless bosonic content comprises the graviton, dilaton, Kalb–Ramond field (NS–NS) together with RR forms of even degree.
- Type IIB is chiral: both spinors have the same chirality. The R–R sector now contains RR forms of odd degree and a self-dual 5-form, while the NS–NS sector remains identical to IIA.

The difference originates in the relative choice of GSO signs between left- and right-movers; only the chiral choice preserves a single ten-dimensional chirality.

Example

The massless graviton arises in the NS–NS sector after GSO projection. The GSO projection discards the tachyon, which is the NS ground state itself.

10.4 Heterotic Strings

A hybrid construction places a left-moving bosonic string (critical dimension 26) alongside a right-moving superstring (critical dimension 10). The sixteen extra left-moving bosons are compactified on an even self-dual Euclidean lattice of rank 16. Two such lattices exist that give the gauge groups SO(32) or $E_8 \times E_8$. The resulting theories therefore carry $\mathcal{N} = 1$ spacetime supersymmetry together with non-Abelian gauge symmetry of rank 16.

Because the lattices are even and self-dual, the one-loop partition function is modular invariant and the gauge and gravitational anomalies cancel by the Green–Schwarz mechanism. The low-energy limit is ten-dimensional $\mathcal{N} = 1$ supergravity coupled to super-Yang–Mills theory with gauge group either SO(32) or $E_8 \times E_8$.

10.5 Type I Theory and Anomaly Cancellation

Starting from Type IIB and quotienting by worldsheet parity reversal yields an unoriented theory containing both open and closed strings. The open-string sector carries Chan–Paton factors transforming in the fundamental of SO(32). Consistency again requires the Green–Schwarz cancellation of anomalies, which occurs precisely for this gauge group. The resulting spectrum matches that of the heterotic SO(32) theory at the massless level, although the two constructions differ non-perturbatively.

All five theories—Type IIA, Type IIB, Type I, heterotic SO(32) and heterotic $E_8 \times E_8$—are therefore free of perturbative anomalies in ten dimensions and reduce at low energy to the corresponding supergravity multiplets.

10.6 Sources

- https://arxiv.org/pdf/2212.06187
- https://arxiv.org/pdf/hep-th/0207142
- https://arxiv.org/pdf/hep-th/0002165
- https://arxiv.org/pdf/0910.2254

10.7 Exercises

Exercise 1. Derive the critical dimension of the RNS superstring by computing the total central charge from the bosonic and fermionic contributions and requiring anomaly cancellation in the super-Virasoro algebra.

Exercise 2. State the boundary conditions on the worldsheet fermions ψ^μ that define the NS and R sectors. Identify the L_0 eigenvalue of each ground state and explain which sector contains a tachyon before the GSO projection.

Exercise 3. Explain how the GSO projection is implemented via worldsheet fermion-number parity and why it simultaneously eliminates the tachyon while enforcing equal numbers of bosons and fermions at every mass level.

Exercise 4. List the massless bosonic fields that survive the GSO projection in the NS–NS sector. Identify which additional massless bosonic fields appear in the R–R sector of Type IIA versus Type IIB.

Exercise 5. For each of the two heterotic string theories, state the rank-16 even self-dual lattice on which the sixteen extra left-moving bosons are compactified and the resulting ten-dimensional gauge group.

Exercise 6. Describe the worldsheet parity quotient that produces the Type I theory from Type IIB. State the gauge group carried by the open-string Chan–Paton factors that yields a consistent, anomaly-free spectrum.

Exercise 7. Synthesize the five consistent ten-dimensional string theories by tabulating, for each theory, the spacetime supersymmetry, the presence or absence of open strings, and the non-Abelian gauge group (if any).

Exercise 8. Explain why the Green–Schwarz mechanism cancels both gauge and gravitational anomalies in every one of the five theories, and identify the common low-energy effective theory that each string theory reduces to at the massless level.

10.8 Solutions

Solution to Exercise 1.

Each of the ten bosonic coordinates X^μ contributes central charge $c = 1$. Each of the ten Majorana–Weyl fermions ψ^μ contributes $c = \frac{1}{2}$. The total central charge is therefore $c = 10 + 5 = 15$. The super-Virasoro algebra is free of anomalies only when $c = 15$, which forces the critical dimension to be $D = 10$.

Solution to Exercise 2.

In the NS sector the fermions obey antiperiodic boundary conditions $\psi^{\mu(\sigma+2\pi)} = -\psi^{\mu(\sigma)}$, so the modes of G_r are half-integer and the ground-state L_0 eigenvalue is $-\frac{1}{2}$, producing a tachyon. In the R sector the fermions are periodic, the modes of G_r are integer, and the ground state is massless but degenerate, realizing a representation of the zero-mode Clifford algebra.

Solution to Exercise 3.

The GSO projection retains only states with definite worldsheet fermion-number parity $(-1)^F = +1$ (or a fixed overall sign). Because the NS tachyon is odd under this parity it is removed. The surviving spectrum pairs bosons and fermions at every mass level, which is the hallmark of spacetime supersymmetry. The two independent choices of relative sign between left- and right-movers generate the distinct consistent theories.

Solution to Exercise 4.

The NS–NS sector yields the graviton, dilaton and Kalb–Ramond two-form in both Type IIA and Type IIB. The R–R sector of Type IIA contains even-degree forms (1-form and 3-form), while Type IIB contains odd-degree forms (0-form, 2-form and self-dual 4-form, together with the self-dual 5-form).

Solution to Exercise 5.

The two even self-dual Euclidean lattices of rank 16 are the root lattice of SO(32) and the direct sum of the two E_8 root lattices. Compactification on the former produces the heterotic SO(32) theory; compactification on the latter produces the heterotic $E_8 \times E_8$ theory.

Solution to Exercise 6.

Type I theory is obtained by quotienting Type IIB by worldsheet parity reversal, which introduces unoriented surfaces and open strings. Consistency of the resulting spectrum under the Green–Schwarz mechanism requires the open-string gauge group to be SO(32).

Solution to Exercise 7.

Type IIA: $\mathcal{N} = 2$ SUSY, closed oriented strings only, no gauge group. Type IIB: $\mathcal{N} = 2$ SUSY, closed oriented strings only, no gauge group. Type I: $\mathcal{N} = 1$ SUSY, open and closed unoriented strings, gauge group SO(32). Heterotic SO(32): $\mathcal{N} = 1$ SUSY, closed oriented strings, gauge group SO(32). Heterotic $E_8 \times E_8$: $\mathcal{N} = 1$ SUSY, closed oriented strings, gauge group $E_8 \times E_8$.

Solution to Exercise 8.

Each of the five theories possesses a modular-invariant one-loop partition function (guaranteed by the choice of even self-dual lattices or by the orientifold projection). Modular invariance together with the Green–Schwarz cancellation mechanism removes all gauge and gravitational anomalies. The massless spectrum of every theory is precisely that of the corresponding ten-dimensional supergravity multiplet (Type IIA or IIB supergravity, or $\mathcal{N} = 1$ supergravity coupled to SO(32) or $E_8 \times E_8$ super-Yang–Mills).

11 Dualities, M-Theory, and 11-Dimensional Supergravity

11.1 The Web of String Dualities

All five consistent superstring theories arise as different perturbative expansions of a single underlying structure. T-duality inverts the radius of a compact spatial circle, $R \to \frac{\alpha'}{R}$, while exchanging momentum modes with winding modes. This maps Type IIA to Type IIB on a circle and likewise identifies the two heterotic theories. S-duality inverts the string coupling, $g_s \to \frac{1}{g_s}$, thereby exchanging strong and weak coupling regimes. U-duality is generated by the simultaneous action of T- and S-dualities and therefore acts on the full lattice of charges and moduli that survive compactification.

Key Idea

The five perturbative string theories are not independent; each is a different weak-coupling description of the same non-perturbative theory. Tracking how T-, S-, and U-dualities act on radii, couplings, and brane charges reveals the connectivity of the web.

11.2 Strong Coupling and the Emergence of M-Theory

When the Type IIA string coupling g_s is taken to infinity, an extra spatial dimension decompactifies. The radius of this eleventh dimension grows linearly with g_s, $R_{11} = g_s\sqrt{\alpha'}$. In this limit the D0-branes of Type IIA become Kaluza–Klein momentum modes along the new circle. The resulting eleven-dimensional theory is M-theory; its low-energy effective description is eleven-dimensional supergravity. All five string theories are recovered from M-theory by suitable compactifications and limits, with membrane/fivebrane duality descending to the known string/string dualities upon dimensional reduction.

Definition — M-theory

An eleven-dimensional theory whose perturbative expansions include the five consistent superstring theories. It is defined by its low-energy supergravity limit together with non-perturbative M2- and M5-branes; it is not itself a conventional string theory.

11.3 Eleven-Dimensional Supergravity and M-Branes

The bosonic sector of eleven-dimensional supergravity contains the metric g_{MN} and a three-form potential C_3 whose field strength is $G_4 = dC_3$. The Chern–Simons term $\int C_3$ wedge G_4 wedge G_4 is required by supersymmetry. The theory admits two fundamental extended objects: the M2-brane, which couples electrically to C_3, and the M5-brane, which couples magnetically. Their tension ratio is fixed by the eleven-dimensional Planck length. Dimensional reduction of the M2-brane on a circle yields the Type IIA fundamental string, while reduction of the M5-brane yields the NS5-brane or D4-brane depending on the orientation of the circle.

Example

A stack of N coincident M2-branes preserves 16 supercharges. The near-horizon geometry is $\mathrm{AdS}_4 \times S^7$ with radius proportional to $N^{\frac{1}{6}}\ell_p$. Upon reduction along a transverse circle this solution descends to the $\mathrm{AdS}_4 \times \mathrm{CP}^3$ background of massless Type IIA, furnishing an explicit M-theory lift of a known string-theory solution.

11.4 Type IIB S-Duality and SL(2,Z)

Type IIB string theory is invariant under an SL(2,Z) transformation that acts on the axio-dilaton $\tau = C_0 + ie^{-\Phi}$. The two three-form field strengths transform as a doublet, so that NS-NS and R-R fluxes are interchanged when $\tau \to -\frac{1}{\tau}$. Because the transformation is an exact symmetry of the quantum theory, the self-duality is non-perturbative. In lower dimensions this SL(2,Z) enlarges to the U-duality group that acts on the full charge lattice.

Theorem — Type IIB self-duality

The transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$ leaves the Type IIB equations of motion and the spectrum of D-branes invariant. In particular the D1-brane is mapped to the fundamental string when $g_s \to \frac{1}{g_s}$.

11.5 Heterotic–Type I Duality

Heterotic SO(32) string theory at strong coupling is dual to Type I string theory at weak coupling. The heterotic fundamental string is mapped to the Type I D1-brane, while the heterotic NS5-brane is identified with the Type I D5-brane. Anomaly cancellation proceeds via anomaly inflow: the chiral fermions living on the D1-brane world-volume cancel the bulk anomaly that would otherwise appear in the ten-dimensional theory. Matching of the spectra, the low-energy effective actions, and the BPS mass formulae confirms the duality.

Key Idea

D-branes and NS5-branes are non-perturbative solitons whose tensions scale as $\frac{1}{g_s}$ or $\frac{1}{g_s^2}$. Their existence makes the strong-coupling limits of each string theory accessible and demonstrates that the five theories are different windows onto one underlying structure.

11.6 Sources

- https://ar5iv.labs.arxiv.org/html/hep-th/9609051
- https://ar5iv.labs.arxiv.org/html/hep-th/9608117
- https://ar5iv.labs.arxiv.org/html/hep-th/9903165
- https://arxiv.org/abs/hep-th/9410237
- https://arxiv.org/abs/1306.2643

11.7 Exercises

Exercise 1. Explain how T-duality acts on the radius of a compact circle and why it maps Type IIA string theory to Type IIB string theory.

Exercise 2. State the relation between the Type IIA string coupling g_s and the radius R_{11} of the eleventh dimension that emerges at strong coupling. Identify the resulting theory.

Exercise 3. Describe the action of the SL(2,Z) duality group on the axio-dilaton $\tau = C_0 + ie^{-\Phi}$ of Type IIB string theory and state what it implies for NS-NS and R-R three-form fluxes.

Exercise 4. The bosonic fields of eleven-dimensional supergravity consist of the metric g_{MN} and the three-form potential C_3. Write the field strength G_4 and the Chern–Simons term required by supersymmetry. Identify which branes couple electrically and magnetically to C_3.

Exercise 5. An M2-brane reduced on a transverse circle yields a fundamental string of Type IIA. An M5-brane reduced on a circle yields either an NS5-brane or a D4-brane. Explain the dependence on the orientation of the reduction circle and how this is consistent with the web of dualities.

Exercise 6. Under the heterotic–Type I duality the heterotic fundamental string is identified with the Type I D1-brane. Using the scaling of tensions with the string coupling, show that this identification is consistent with the inversion $g_s \to \frac{1}{g_s}$.

Exercise 7. Consider a stack of N coincident M2-branes whose near-horizon geometry is $\mathrm{AdS}_4 \times S^7$. After reduction along a transverse circle the geometry becomes $\mathrm{AdS}_4 \times \mathrm{CP}^3$. Verify that the radius scaling $N^{\frac{1}{6}}\ell_p$ is compatible with the known massless Type IIA background and state how many supercharges are preserved.

Exercise 8. Using only the T- and S-duality rules given in the module, construct a closed chain of dualities that connects the strong-coupling limit of Type IIA to the weak-coupling limit of Type IIB via an intermediate M-theory configuration. Indicate which branes become which under each step.

11.8 Solutions

Solution to Exercise 1.

T-duality inverts the radius of a compact spatial circle according to $R \to \frac{\alpha'}{R}$. Momentum modes $p = \frac{n}{R}$ are thereby exchanged with winding modes $w = m\frac{R}{\alpha'}$. Because the two chiralities of the world-sheet fermions transform differently under this exchange, the operation maps the opposite-chirality theory (Type IIA) to the same-chirality theory (Type IIB) and vice versa.

Solution to Exercise 2.

When the Type IIA coupling g_s is sent to infinity the radius of the eleventh dimension grows as $R_{11} = g_s\sqrt{\alpha'}$. The D0-branes become Kaluza–Klein modes along this circle. The resulting eleven-dimensional theory is M-theory, whose low-energy limit is eleven-dimensional supergravity.

Solution to Exercise 3.

The axio-dilaton transforms as the fractional linear transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. The two three-form field strengths form an SL(2,Z) doublet; consequently an NS-NS flux is mapped into an R-R flux (and vice versa) whenever $\tau \to -\frac{1}{\tau}$. Because the transformation is an exact quantum symmetry, the spectrum of D-branes remains invariant.

Solution to Exercise 4.

The four-form field strength is $G_4 = dC_3$. Supersymmetry requires the Chern–Simons interaction $\int C_3$ wedge G_4 wedge G_4. The M2-brane couples electrically to C_3 while the M5-brane couples magnetically to the dual six-form potential whose field strength is $* G_4$.

Solution to Exercise 5.

Reduction along a circle longitudinal to an M5-brane yields the NS5-brane of Type IIA; reduction along a transverse circle yields the D4-brane. The M2-brane reduced along a transverse circle produces the fundamental Type IIA string. These identifications are required by the matching of tensions and by the fact that T-duality interchanges the two heterotic theories while S-duality interchanges strong and weak coupling.

Solution to Exercise 6.

The tension of the heterotic fundamental string scales as $T_{\mathrm{F1}} \sim \frac{1}{g_s^0}$, while the tension of the Type I D1-brane scales as $T_{\mathrm{D1}} \sim \frac{1}{g_s}$. Under the duality map $g_s \to \frac{1}{g_s}$ the two tensions are interchanged, establishing that the heterotic string at strong coupling is identified with the Type I D-string at weak coupling.

Solution to Exercise 7.

The near-horizon radius of the M2-brane solution scales as $R \sim N^{\frac{1}{6}} \ell_p$. Upon reduction on a transverse circle the eleven-dimensional Planck length and the Type IIA string length are related by the usual Kaluza–Klein dictionary; the resulting Type IIA radius reproduces the known $\mathrm{AdS}_4 \times \mathrm{CP}^3$ background. Both geometries preserve 16 supercharges, matching the amount preserved by a stack of M2-branes.

Solution to Exercise 8.

Start with Type IIA at strong coupling; this is M-theory on a large circle. Compactify M-theory further on a second circle and perform an M-theory T-duality (membrane/fivebrane exchange) to reach Type IIB on a circle of inverted radius. Finally apply an S-duality transformation of Type IIB to reach weak coupling. The composite operation closes the duality web and maps every brane charge to its dual counterpart, confirming that all five perturbative string theories are different weak-coupling limits of a single underlying structure.

12 Compactifications, Calabi-Yau Manifolds, Phenomenology, Black Holes, and AdS/CFT

12.1 Calabi-Yau Manifolds and Holonomy

String theory in ten dimensions must be reduced to four observable dimensions while preserving some supersymmetry. The internal six-dimensional space must admit a Ricci-flat metric with reduced holonomy. A Calabi-Yau threefold satisfies these requirements: it is a complex Kähler manifold whose first Chern class vanishes, guaranteeing, by Yau's theorem, a unique Ricci-flat Kähler metric in each Kähler class.

The holonomy group of such a manifold is contained in $SU(3)$. Parallel transport of a spinor around any closed loop therefore leaves a covariantly constant spinor invariant. This single invariant spinor generates the four unbroken supercharges of an $N = 1$ theory in four dimensions when the heterotic string is compactified, or the eight supercharges of an $N = 2$ theory when the Type II string is compactified.

Definition — Calabi-Yau threefold

A compact complex threefold X with trivial canonical bundle $K_X = \mathrm{wedge}^3 T^*X$ and $h^{1,0}(X) = 0$. Equivalently, a Ricci-flat Kähler manifold with $SU(3)$ holonomy.

The Hodge numbers $h^{1,1}$ and $h^{2,1}$ count the Kähler and complex-structure moduli, respectively. Each modulus appears in the four-dimensional effective theory as a massless scalar field.

12.2 Heterotic and Type II Compactifications

When the heterotic string is placed on a Calabi-Yau threefold, the ten-dimensional gauge group $E_8 \times E_8$ is broken by the embedding of the $SU(3)$ spin connection into one E_8 factor. The resulting four-dimensional gauge group is $E_6 \times E_{8'}$, with chiral matter in the 27 and $\overline{27}$ representations of E_6. The net number of generations is given by the topological index $N_{\text{gen}} = \frac{1}{2}|\chi(X)| = \left|h^{1,1} - h^{2,1}\right|$.

Type II strings on the same threefold produce $N = 2$ supergravity. In Type IIB string theory compactified on a Calabi-Yau threefold, the vector multiplets are parametrized by the complex-structure moduli space of dimension $h^{2,1}$, while the hypermultiplets are parametrized by the Kähler moduli together with the dilaton-axion. (In Type IIA the assignment of $h^{1,1}$ and $h^{2,1}$ is interchanged.) Mirror symmetry exchanges these two moduli spaces, furnishing a non-perturbative equivalence between Type IIA and Type IIB compactifications.

Key Idea

The massless spectrum is completely determined by the topology of the Calabi-Yau threefold. Gauge groups, chiral matter, and the number of moduli follow directly from the Hodge numbers and the embedding of the spin connection.

Yukawa couplings among chiral fields arise from triple intersections of homology cycles. In the Type IIA picture these are classical intersection numbers; in the Type IIB picture they receive world-sheet instanton corrections that are summed by the mirror map.

12.3 Extremal Black Holes and Microstate Counting

The Strominger-Vafa construction realizes certain extremal black holes in Type II string theory as bound states of D-branes wrapped on cycles of $K3 \times S^1$, yielding five-dimensional black holes. The Bekenstein-Hawking entropy is

$$S_{\mathrm{BH}} = \frac{A}{4G_N},$$

where A is the horizon area. On the microscopic side the same quantity is reproduced by counting the degeneracy of open-string states living on the D-brane world-volume.

For a stack of D4-branes wrapping a four-cycle P together with D0-branes, the number of microstates is given by the Witten index of the resulting 0+1-dimensional supersymmetric quantum mechanics. In the limit of large charges the Cardy formula yields

$$S_{\mathrm{micro}} = 2\pi\sqrt{N_1 N_2 N_3 - J^2},$$

(for appropriate 5D three-charge black holes), matching the macroscopic area law. This agreement constitutes the first microscopic derivation of black-hole entropy in a controlled quantum theory of gravity.

Theorem — Strominger-Vafa entropy match

For BPS black holes carrying D-brane charges on a Calabi-Yau threefold, the asymptotic degeneracy of microstates equals the Bekenstein-Hawking entropy in the large-charge limit.

12.4 The AdS/CFT Correspondence

A stack of N coincident D3-branes in Type IIB string theory sources the geometry

$$\mathrm{ds}^2 = H^{-\frac{1}{2}}(-dt^2 + dx^2) + H^{\frac{1}{2}} dr^2 + r^2 d\Omega_5^2,$$

with $H = 1 + 4\pi g_s N \frac{(\alpha')^2}{r^4}$. In the near-horizon limit $r \to 0$ the geometry factorizes into $AdS_5 \times S^5$. The same limit decouples the open-string dynamics on the branes from the closed strings in flat space, leaving $N = 4$ super-Yang-Mills theory with gauge group $SU(N)$ on the boundary.

The duality identifies single-trace operators in the gauge theory with bulk fields. The stress-tensor multiplet, for example, maps to the graviton and its superpartners. Correlation functions of these operators are computed by evaluating the on-shell supergravity action subject to prescribed boundary conditions. The expectation value of a circular Wilson loop is given by the (regularized) area of the minimal string world-sheet ending on the loop, reproducing the strong-coupling result

$$\langle W \rangle \sim \left(\sqrt{\lambda}\right)^{-\frac{1}{2}} \exp\left(+\sqrt{\lambda}\right),$$

(for the ½-BPS case), where $\lambda = g_{\mathrm{YM}}^2 N$ is the ‘t Hooft coupling.

Example

In the large-N limit with λ fixed, the gravitational side becomes classical Type IIB supergravity on $AdS_5 \times S^5$. All planar correlation functions of $N = 4$ SYM are thereby obtained from tree-level Witten diagrams.

12.5 The Landscape and Phenomenological Challenges

Realistic models must simultaneously achieve chiral matter, anomaly cancellation, moduli stabilization, and a small positive cosmological constant. Flux compactifications generate a superpotential that lifts many moduli, yet the resulting vacuum energy landscape contains an exponentially large number of vacua. Systematic searches therefore rely on statistical techniques rather than exhaustive enumeration.

Heterotic orbifold constructions have produced spectra reproducing MSSM features, but realistic Yukawa textures remain an open problem. Persistent open problems include the dynamical selection of the correct vacuum, the suppression of proton decay, and the incorporation of neutrino masses while maintaining moduli stabilization.

Key Idea

String theory supplies a vast landscape of four-dimensional effective theories. Only a subset can be rendered consistent with all phenomenological constraints; identifying that subset remains an active area of research.

12.6 Exercises

Exercise 1. State the holonomy group of a Calabi-Yau threefold and the number of unbroken supercharges obtained upon compactifying the heterotic string on such a manifold.

Exercise 2. A Calabi-Yau threefold has Hodge numbers $h^{1,1} = 3$ and $h^{2,1} = 27$. Compute the net number of generations of chiral matter in a heterotic $E_8 \times E_8$ compactification on this manifold.

Exercise 3. In a Type IIB compactification on a Calabi-Yau threefold, identify which moduli space is parametrized by the vector multiplets and which by the hypermultiplets. How does mirror symmetry interchange these assignments?

Exercise 4. Explain why the Yukawa couplings among chiral multiplets are given by classical triple intersection numbers in Type IIA but receive non-perturbative corrections in Type IIB.

Exercise 5. For a five-dimensional three-charge extremal black hole realized by D-branes on $K3 \times S^1$, the microscopic entropy takes the form $S_{\text{micro}} = 2\pi\sqrt{N_1 N_2 N_3 - J^2}$. Show that this expression reproduces the Bekenstein-Hawking entropy in the large-charge limit.

Exercise 6. Write the near-horizon geometry sourced by N coincident D3-branes and identify the dual four-dimensional gauge theory that arises in the decoupling limit.

Exercise 7. Using the AdS/CFT dictionary, obtain the leading strong-coupling behavior of the expectation value of a circular ½-BPS Wilson loop in $N = 4$ super-Yang-Mills theory from the area of the minimal string world-sheet in $AdS_5 \times S^5$.

Exercise 8. A flux compactification generates an exponentially large number of vacua. List three independent phenomenological requirements that must still be satisfied before any vacuum can be considered realistic, and briefly indicate why each is nontrivial.

12.7 Solutions

Solution to Exercise 1.

A Calabi-Yau threefold is a Ricci-flat Kähler manifold whose holonomy is contained in $SU(3)$. Parallel transport therefore leaves a single covariantly constant spinor invariant. For the heterotic string this spinor generates four unbroken supercharges, corresponding to $N = 1$ supersymmetry in four dimensions.

Solution to Exercise 2.

The Euler characteristic is $\chi(X) = 2(h^{1,1} - h^{2,1}) = 2(3 - 27) = -48$. The net number of generations is therefore $N_{\text{gen}} = \frac{1}{2}|\chi(X)| = 24$.

Solution to Exercise 3.

In Type IIB the vector multiplets are parametrized by the complex-structure moduli (dimension $h^{2,1}$) while the hypermultiplets are parametrized by the Kähler moduli together with the dilaton-axion. Mirror symmetry exchanges $h^{1,1}$ with $h^{2,1}$ and therefore interchanges the two moduli spaces, mapping Type IIB to Type IIA (and vice versa).

Solution to Exercise 4.

In Type IIA the relevant Yukawa couplings arise from classical intersection numbers of three-cycles. In Type IIB the same couplings are realized by world-sheet instantons; these non-perturbative corrections are resummed by the mirror map that relates the two pictures.

Solution to Exercise 5.

In the large-charge regime the Cardy formula for the degeneracy of D-brane microstates yields precisely $S_{\text{micro}} = 2\pi\sqrt{N_1 N_2 N_3 - J^2}$. The horizon area of the corresponding five-dimensional black hole is $A = 8\pi G_5 \sqrt{N_1 N_2 N_3 - J^2}$. Substituting into the Bekenstein-Hawking formula $S_{\text{BH}} = \frac{A}{4G_5}$ reproduces the microscopic expression, establishing the Strominger-Vafa match.

Solution to Exercise 6.

The near-horizon geometry is $AdS_5 \times S^5$ with radius set by N. The decoupled open-string dynamics on the D3-branes become $N = 4$ super-Yang-Mills theory with gauge group $SU(N)$ living on the conformal boundary of AdS_5.

Solution to Exercise 7.

The ½-BPS circular Wilson loop is dual to a fundamental string whose world-sheet is a minimal surface ending on the loop at the boundary. The regularized on-shell area evaluates to $2\pi\sqrt{\lambda}$, where $\lambda = g_{\text{YM}}^2 N$ is the ‘t Hooft coupling. Hence $\langle W \rangle \sim \left(\sqrt{\lambda}\right)^{-\frac{1}{2}} \exp\left(+\sqrt{\lambda}\right)$ at strong coupling.

Solution to Exercise 8.

Any realistic vacuum must (i) contain a chiral spectrum whose gauge group and matter representations embed the Standard Model (or MSSM) while cancelling all anomalies; (ii) stabilize all moduli at values that produce realistic Yukawa couplings and suppress proton decay; (iii) generate a small positive cosmological constant consistent with observation. Each requirement is nontrivial because the flux superpotential, non-perturbative effects, and the enormous number of vacua make systematic selection of a phenomenologically viable point extremely difficult.